A Dynamic Theory of Learning and Relationship Lending

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Abstract

We introduce learning into a banking model to study the dynamics of relationship lending. In our model, an entrepreneur chooses between bank and market financing. Bank lending facilitates learning *over time*, but it subjects the borrower to the downside of information monopoly. We construct an equilibrium in which the entrepreneur starts with bank financing and subsequently switches to the market, and we find conditions under which this equilibrium is unique. Our model generates several novel results: 1) Endogenous zombie lending, i.e. the bank is willing to roll over loans known to be bad for the prospect of future loan sales. 2) Short maturity could encourage zombie lending and deteriorate credit quality; and 3) the information-monopoly cost may increase or decrease with the length of the lending relationship.

Keywords: private learning, experimentation, relationship banking, information monopoly, debt rollover, extend and pretend, adverse selection, dynamic games

1 Introduction

How do lending relationships evolve over time? How do firms choose *dynamically* between bank and market financing? Why do banks sometimes roll over loans that are known to be insolvent? To answer these questions, we introduce a dynamic framework in the context of relationship lending. By doing so, we also examine how the magnitude of information-monopoly cost changes as lending relationship continues.

It has been widely documented that bank loans contain important information about borrowers that is not available to market-wide lenders (Addoum and Murfin, 2017; James, 1987; Gustafson et al., 2017). Moreover, as suggested by Lummer and McConnell (1989), such information is not produced upon a bank's first contact with a borrower, but, instead, through repeated interactions during prolonged lending relationships which involve substantive screening and monitoring. On the other hand, as shown in Rajan (1992), learning provides information advantage to the relationship bank and thus increases the information-monopoly cost so that ultimately, the borrower may switch to lenders in the financial market. When should borrowers switch from relationship lending to market financing? How do entrepreneurs balance the tradeoff between learning and information-monopoly cost? How does loan maturity affect these decisions?

To answer these questions, we introduce private learning into a dynamic model of relationship lending. Specifically, we model an entrepreneur investing in a long-term, illiquid project whose quality is either good or bad. Only a good project has positive net present value (NPV) and should be financed. A bad project should be liquidated immediately. The liquidation value is a constant and independent of the project's quality. Initially, the quality of the project is unknown to anyone, including the entrepreneur herself. She can raise funding from either the competitive financial market or a bank that will develop into a relationship. Market financing takes the form of arm's-length debt so that lenders only need to break even given their beliefs on the project's quality. Under market financing, no information is ever produced and therefore, the maturity of the market debt is irrelevant. In contrast, if the entrepreneur borrows from a bank, screening and monitoring will produce "news" about the project's quality. We model news arrival as a Poisson event and assume this news is only observed by the entrepreneur and the bank once the relationship starts. In other words, the bank and the entrepreneur learn *privately* about the project's quality as time goes by. Meanwhile, all agents, including lenders in the financial market can observe the time since the initialization of the project, which will turn out to be the state variable.

Given the structure of learning, the bank and the borrower possess one of the three types of private information after time 0: 1) news has arrived and implies the project is good – the

informed-good type g; 2) news has arrived and implies the project is bad – the informed-bad type b; and 3) no news has arrived yet – the uninformed type u. Upon the maturity of the bank loan, the bank and the entrepreneur jointly determine whether to roll it over, to liquidate the project, or to switch to market-based financing. In the case of rollover, the price of the loan is determined by Nash Bargaining between the bank and the entrepreneur.

By solving the model in closed form, we characterize the equilibrium with two thresholds $\{t_g, t_b\}$ in the time since project initialization. Consequently, the equilibrium is characterized into three stages. If the bank loan matures between 0 and t_b, an informed-bad type's project will be liquidated. All other types' matured loans will be rolled over. During this period, the average quality of borrowers who remained with banks drifts up because the informed-bad types get liquidated and exit funding. These liquidation decisions are socially efficient and therefore we name this stage after *efficient liquidation*. If the bank loan matures between t_b and t_g, however, it will be rolled over irrespective of the quality of the project. In particular, the relationship bank will roll over the loan matured between t_b and t_g even if bad news has arrived. Clearly, this rollover decision is inefficient. This result on banks' rolling over bad loans can be interpreted as *zombie lending*. Finally, after time passes t_g, all entrepreneurs will switch to market financing upon their bank loans maturing – the *market financing* stage.

The intuitions for these results can be best explained backwards in time. When time elapsed gets sufficiently long, all entrepreneurs will ultimately switch to market financing, driven by the assumption that market-based lenders are competitive and have lower costs of capital. This effect is captured by the threshold t_g. Now, imagine a scenario that bad news arrives shortly before t_g, the relationship-bank could liquidate the project, in which case it receives a fixed payoff. Alternatively, it can rollover the loan and pretend as if no bad news has arrived yet. Essentially, by hiding losses *today*, the bank helps the borrower accumulate reputation so that the loan could be sold to the market in the *future*. Such "extending and pretending" incur relatively low costs since shortly afterwards, these bad loans will be sold to the lenders in the market, and the part of the loss will be shared. On the other hand, if negative news arrives early on, "extending and pretending" are much more costly, due to both large time discounting and the high probability that before t_g, the project may mature and the loss will be entirely born by the relationship bank. In this case, liquidating the project is the more profitable option. The threshold t_b captures the time at which an informed-bad type is indifferent between liquidating and rolling over. Note that there is a significant gap between t_b and t_g so that the zombie lending stage lasts for a significant period. During this period, the average quality of borrowers stays unchanged. However, this period is necessary to force informed-bad types to liquidate and exit before t_b, which leads to an improvement in the average quality during the efficient liquidation stage.

We show that short-term loan leads to a longer-period of zombie lending and reduces the credit quality that is ultimately financed by the market. This result is in contrast with previous studies, which show debt with shorter maturities can better align incentives across different parties (Diamond, 1991a). Intuitively, under shorter maturity the loan can be sold faster once the market financing stage arrives. As a result, the benefits to "extend and pretend" get higher so that fewer of the informed-bad types liquidate their projects before t_b, deteriorating the credit quality.

We show the magnitude of information-monopoly cost, proxied by the continuation payoff of the entrepreneur, can be non-monotonic in the length of the relationship. This pattern is especially prominent for the informed-bad type if 1) loan maturity is short, 2) the entrepreneur's bargaining power is high, and 3) the project's liquidation value is low. Intuitively, two effects are at work here. First, as time approaches the market financing stage, the value of a bad project increases and so is the surplus from rolling over a bad loan. Ceteris paribus, the entrepreneur's continuation payoff should increase. However, there is a second, counter-veiling effect. In the zombie lending stage, the bank's outside option in Nash Bargaining is to liquidate the project which only generates a (relatively) low value. In this case, the entrepreneur is essentially "holding up" the bank. By contrast, during the market financing stage the bank will be very likely to recover the full value of the loan. The ability for the entrepreneur to hold up the bank then gets more and more limited as the time gets closer and closer to the market financing stage. Ceteris paribus, the entrepreneur's continuation payoff should decrease. The overall pattern therefore depends on the relatively magnitude of these two effects.

Related Literature

Finally, our modeling approach builds on the literature on private, learning, reputation and experimentation (Che and Hörner, 2017; Akcigit and Liu, 2015; Kremer et al., 2014; Grenadier et al., 2014; Martel et al., 2018; Daley and Green, 2012). Our paper is closely related to Hwang (2018), except for the assumptions on gains from trade and a competitive set of buyers (market-based lenders in our mode), which drive the crucial difference in equilibrium dynamics. Our paper is among the first set of papers in the context of banking (also see Halac and Kremer (2018)).

Our paper extends the literature to study the dynamics of relationship lending (Diamond, 1991b; Rajan, 1992). In Diamond (1991b), the lender's decision is mypoic because borrowers' projects mature after one period. Therefore, a lender would never want to engage in zombie lending. Rajan (1992) studies the tradeoff between relationship-based lending and

arm's length debt. It implies that the information-monopoly problem increases over time as the bank gets more and more informative. However, if the relationship-bank keeps rolling over, it is good news to the market, which will lead highly-reputable borrowers switch to market finance. Our paper explicitly studies such a switch and examine how the information-monopoly cost varies as the lending relationship continues. Our paper is also related to Parlour and Plantin (2008), which study the efficiency of a secondary market for loan sales.[1] Our paper is also related to a set of papers that adopt the Leland-type framework to study firm's decisions to rollover debt (He and Xiong, 2012; He and Milbradt, 2016). The existing literature has mostly focused on a set of competitive financiers without private information.

Existing explanations on zombie lending largely rely on either loan officers' career concerns (Rajan, 1994) or additional regulatory capital triggered by writing off bad loans (Caballero et al., 2008; Peek and Rosengren, 2005). We offer a dynamic explanation based on the prospect of future loan sales. This result is also related to Kremer and Skrzypacz (2007) and Fuchs and Skrzypacz (2015) which study how suspension and delaying trading can promote efficiency in markets plagued by adverse selection.

2 Model

We consider a continuous-time model with an infinite horizon. An entrepreneur (she) invests in a long-term project with unknown quality. She borrows from either a bank that will develop into a relationship or the competitive financial market. Compared to market financing, bank financing has the advantage of producing valuable information but with the downside of a higher cost of capital and the possibility of information monopoly. Below, we describe the model in detail.

2.1 Project

We consider a long-term project that generates a constant stream of interim cash flows cdt over a period $[t, t + dt]$. The project matures at a random time τ_ϕ, which arrives at an exponential time with intensity $\phi > 0$. Upon maturity, the project produces some random final cash flows $\tilde{R}$, depending on its type. A good (g) project produces cash flows $\tilde{R} = R$ with certainty, whereas a bad (b) project produces $\tilde{R} = R$ with probability $\theta < 1$. With probability $1 - \theta$, a matured bad project fails to produce any final cash flows, i.e., $\tilde{R} = 0$. Initially, no agent, including the entrepreneur herself knows the exact type of the project;

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[1]There is also an empirical literature on loan sales and lending relationships (Drucker and Puri, 2008), market reactions to loan sales (Dahiya et al., 2003).

all agents share the public belief that q_0 is the probability of the project's type being good. At any time before the final cash flows are produced, the project can be terminated with a liquidation value $L > 0$. Note that the liquidation value is independent of the project's quality, so it shall be understood as the liquidation of the physical asset used in production. Let $r > 0$ be the entrepreneur's discount rate and therefore the fundamental value of the project to the entrepreneur is given by the discounted value of its future cash flows:

$$NPV_r^g = \frac{c + \phi R}{r + \phi}, \quad NPV_r^b = \frac{c + \phi \theta R}{r + \phi}, \quad NPV_r^u = q_0 NPV_r^g + (1 - q_0) NPV_r^b. \quad (1)$$

2.2 Agents and Debt Financing

The borrower has no initial wealth and needs to finance the entire investment outlay by borrowing through debt contracts.[2] The use of debt contracts can be justified by non-verifiable final cash flows (Townsend, 1979). One can therefore think of the entrepreneur as a manager of a start-up venture who faces financial constraints. We consider two types of debt, offered by banks and market-based lenders, respectively. First, the entrepreneur can take out a loan from a banker (he), who has the same discount rate r. Following Leland (1998), we assume a bank loan lasts for a random period and matures at a random time τ_m, upon the arrival of an independent Poisson event with intensity $m > 0$. The assumption of exponentially maturing loan simplifies the analysis, since at any time before the loan matures, the expected remaining maturity is always $\frac{1}{m}$. In subsection 4.2, we study the case with deterministic maturity and show all the results carry over.

The second type of debt is provided by the market and thus can be thought as public bonds. In particular, we consider a competitive financial market in which lenders have discount rate δ satisfying $\delta \in (0, r)$. As a result, market financing is cheaper than bank financing so that if the project's type were publicly known, the entrepreneur would strictly prefer to borrow from the market. Relate to (1), let us define the NPV of the project to the market as

$$NPV_\delta^g = \frac{c + \phi R}{\delta + \phi}, \quad NPV_\delta^b = \frac{c + \phi \theta R}{\delta + \phi}, \quad NPV_\delta^u = q_0 NPV_\delta^g + (1 - q_0) NPV_\delta^b. \quad (2)$$

The assumption $\delta < r$ captures the realistic feature that banks have higher cost of capital than the market, which can be justified by either regulatory requirements or the skin in the game needed to monitor borrowers (see Holmstrom and Tirole (1997) for example).[3] As it

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[2]We will derive the maximum amount that she can raise at the initial data after solving the model, in which case we can discuss the minimum net worth needed to finance a project with a fixed investment scale.

[3]The entire model can be written as one with $r = \delta$ but there is transaction cost associated with rolling

will be clear shortly, the maturity of the public debt has no effect on the equilibrium outcome and for simplicity, we assume it only matures with the project.

We assume without loss of generality that at any time t, the entrepreneur can only take one type of debt. Both types of debt share the same exogenously-specified face value: $F \in (L, R)$. $F > L$ guarantees that debt is risky, whereas $F < R$ is the maximum pledgeable cash flow.[4] Our paper intends to study the tradeoff between relationship borrowing and public debt, rather than the optimal leverage. At $t = 0$, the entrepreneur chooses between public debt and a bank loan that will develop into a relationship. Once the bank loan matures at τ_m, she can still replace it with public bond. Alternatively, she could roll over the loan with the same bank who may have information monopoly over the project's quality.[5] In this case, the two parties bargain over y_{τ_m}, the coupon rate of the loan that is prevalent from τ_m until the next rollover date. Specifically, we follow Rajan (1992) and model the determinant of y_{τ_m} as a Nash Bargaining game with $(\beta, 1 - \beta)$ being the entrepreneur's and the bank's bargaining power. Note that the entrepreneur cannot promise any coupon payments y_{τ_m} that is above c, the maximum level of the interim cash flow. As we will show shortly, this constraint limits the size of the transfer that the entrepreneur can make to the bank at rollover dates and therefore, the Nash bargaining outcome is sometimes not the one that maximizes the joint surplus of the two parties.

Since market financing is competitive and market-based lenders have a lower cost of capital, the entrepreneur will always prefer to take as high leverage as possible. Therefore, the coupon payments associated with the public bond are assumed as cdt without loss of generality.

2.3 Learning and Information Structure

The quality of the project is initially unknown, with $q_0 \in (0, 1)$ being the belief that it is good. This belief is based on public information and is commonly shared by all agents in the economy. If the entrepreneur finances with the bank, i.e., if she takes out a loan, the entrepreneur-bank pair can *privately* learn the true quality of the project through "news". News arrives at a random time τ_λ, modeled as an independent Poisson event with intensity $\lambda > 0$. Upon arrival, the news fully reveals the true type of the project. In practice, one can think of the news process as information learned during bank screening and monitoring,

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over bank loans.

[4]The maximum pledgeable cash flow can be microfounded by some unobservable action taken by the entrepreneur (e.g. cash diversion) shortly before the final cash flows are produced (Tirole, 2010).

[5]We assume without loss of generality that the entrepreneur would never want to switch to a different banks upon the loan matures. Intuitively, the market has lower cost of capital as an outsider bank and they have the same information structure.

which includes due diligence and covenant violations. We assume that such news can only be observed by the two parties and there is no committable mechanism to share it with third parties such as credit bureaus and market participants. In this sense, the news can be understood as soft information on project quality (Petersen, 2004). For instance, one can think of this as the information that banks acquire upon covenant violation, which includes details on the business prospect, collateral quality, and financial soundness of the borrower.

Remark. Note that learning and news arrival require joint input from both the entrepreneur and the bank. Therefore, we can think of learning as exploration and understanding of the underlying business prospect which require the entrepreneur's experimentation and the bank's previous experience in financing related businesses. In this regard, our banks could also be thought as venture capital firms. Alternatively, we can model learning as a process that solely relies on the entrepreneur's input, whereas the bank simply observes the news content through monitoring. Put it differently, even without bank financing, the entrepreneur will still be able to learn news about the quality of her project over time. Our results are identical in this alternative setting.

Although the public market participants do not observe the news, they can observe t – the project's time since initialization and therefore make inference about the project's quality. Clearly, they form beliefs based on the time elapsed, as well as the decisions during the (random) rollover events. In the benchmark model, we assume the realization of each rollover event τ_m is unobservable to market participants. In subsection 4.2, we relax this assumption and show all results continue to hold qualitatively if rollover events are instead observable. We denote the type of the bank/entrepreneur by $i \in \{u, g, b\}$, where u, g, and b refer to the uninformed, informed-good and informed-bad types, respectively. We assume that that any failure to rollover the debt is publicly observable because in this case either the project will be liquidated or the entrepreneur will switch to market financing. In other words, the market cannot observe when the bank loan has been refinanced with the same bank but can observe whether the firm still has bank loans on its balance sheet. Throughout the paper, we assume the loan contract signed between the bank and the entrepreneur, in particular, the loan rate y_t is not observable by the third party. Therefore, one should interpret y_t not just as the interest rate payments made by the entrepreneur, but also include fees, administrative costs etc.

Given the unique feature of Poisson learning, the *private* belief process, i.e., the belief held by the bank and the entrepreneur, is straightforward. If news hasn't arrived yet, the belief remains at $\mu_t^u = q_0$. In this case, no news is simply no news. Upon news arrival at t_λ, the private belief jumps to $\mu_t^g = 1$ in the case of good news and $\mu_t^b = \theta$ if bad. For the remainder of this paper, we will suppress the time subscripts for private belief and simply use

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$\{\mu^u, \mu^g, \mu^b\}$ without loss of generality. To characterize the public belief process, we introduce a belief system $\{\pi_t^u, \pi_t^g, \pi_t^b\}$, where π_t^u is the public's belief at time t that news hasn't arrived yet, and π_t^g (π_t^b) is the public belief that the news has arrived and is good (bad). In any equilibrium where the belief is rational, π_t^i is consistent with the actual probability that the bank and the entrepreneur are of type $i \in \{u, g, b\}$. Given $\{\pi_t^u, \pi_t^g, \pi_t^b\}$, the public belief that the project is good is given by

$$q_t = \pi_t^u q_0 + \pi_t^g.^6 \tag{3}$$

2.4 Rollover

When the loan matures at τ_m, the entrepreneur and the bank have a total of three options. They can liquidate the project for L, switch to market financing, or continue the relationship by rolling the loan over. Let $O_{\tau_m}^i \equiv O_{E\tau_m}^i + O_{B\tau_m}^i$, $i \in \{u, g, b\}$ be the maximum joint surplus to the two parties in the case that the loan is not rolled over, where $O_{E\tau_m}^i$ and $O_{B\tau_m}^i$ are the value accrued to the entrepreneur and the bank, respectively. Since $F > L$, in the case of liquidation, the bank receives the entire liquidation value L and the entrepreneur receives nothing, i.e., $O_{B\tau_m}^i = L$, and $O_{E\tau_m}^i = 0$. If the two parties choose to switch to market financing, the bank receives full payment $O_{B\tau_m}^i = F$, whereas the entrepreneur receives the remaining surplus $O_{E\tau_m}^i = \bar{V}_{\tau_m}^i - F$, where

$$\bar{V}_{\tau_m}^i = D_{\tau_m} + \frac{\phi \mu^i (R - F)}{r + \phi}. \tag{4}$$

Note that in (4),

$$D_{\tau_m} = \frac{c + \phi [q_{\tau_m} + (1 - q_{\tau_m})\theta] F}{\delta + \phi}, \tag{5}$$

is the amount of proceeds that the entrepreneur raises from the market by issuing a bond with coupon cdt and face value F due whenever the project matures. The two components, $\frac{c}{\delta + \phi}$ and $\frac{\phi[q_{\tau_m} + (1 - q_{\tau_m})\theta]F}{\delta + \phi}$ correspond to the present value of the coupon payments and final payoff, respectively. The second term in (4) is the expected final cash flows that the entrepreneur receives upon the project matures. Because the entrepreneur is financially constrained, the bond price D_{τ_m} must be at least F, implying that

$$q_{\tau_m} \geq q_{\min} \equiv 1 - \frac{c - \delta F}{\phi F (1 - \theta)}. \tag{6}$$

If the entrepreneur and the bank decide to roll over the loan, the two parties bargain over

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[6]To simplify notation, we will abuse notation and use $\{\pi_t^i, q_t\}$ to denote $\{\pi_{t-}^i, q_{t-}\}$. We will state them differently whenever they cause confusions.

the coupon rate y until the next roll-over date. With some abuse of notation, let $B_{\tau_m}(y)$ and $E_{\tau_m}(y)$ be the continuation value for the bank and the entrepreneur if y is the coupon rate decided by the bargaining. [7] Specifically, the Nash Bargaining problem can be written as:

$$y_{\tau_m}^i = \arg\max_{y \leq c} \left\{ \left(B_{\tau_m}^i(y) - O_{B\tau_m}^i\right)^{1-\beta} \left(E_{\tau_m}^i(y) - O_{E\tau_m}^i\right)^{\beta} : B_{\tau_m}^i(y) \geq O_{B\tau_m}^i, E_{\tau_m}^i(y) \geq O_{E\tau_m}^i \right\}. \tag{7}$$

If the solution is interior, that is $y_{\tau_m} < c$, the bank value at the rollover rate is given by the conventional rule for the division of surplus

$$B_{\tau_m}^i = O_{B\tau_m}^i + (1 - \beta)(V_{\tau_m}^i - O_{\tau_m}^i). \tag{8}$$

If the solution is a corner one, i.e., $y_{\tau_m} = c$, the entrepreneur is financially constrained from making a higher transfer to the bank. In this case, $B_{\tau_m}^i < O_{B\tau_m}^i + (1 - \beta)(V_{\tau_m}^i - O_{\tau_m}^i)$. In both cases, it is convenient to write the continuation value of the bank in two parts

$$B_{\tau_m}^i(y) = B_{\tau_m}^i(rF) + T(y), \tag{9}$$

where $B_{\tau_m}^i(rF)$ is the continuation value of the bank with coupon rate rF, and

$$T(y) = \mathbb{E}\left[\int_0^{\tau_m \wedge \tau_\phi} e^{-r(s-t)}(y - rF)\,ds\right] = \frac{y - rF}{r + m + \phi}$$

is the discounted value of all the coupon payments in excess of rF until either the loan or the project matures. Clearly, (9) implies by negotiating the coupon rate, the entrepreneur effectively makes a (possibly negative) transfer to the bank at the rollover date. Note that in principle, Nash Bargaining enables the entrepreneur and the bank to always pursue the option that maximizes their joint surplus. However, this result requires the solution to be implementable by some coupon rate y below c. The financial constraint $y \leq c$ therefore results in scenarios that the maximal joint surplus may not be implementable. Equivalently, the constraint also limits the transfer from the entrepreneur to the bank.

As a result, two conditions must be satisfied for a loan to be rolled over. First, there exists a y such that $B_{\tau_m}^i(y) + E_{\tau_m}^i(y) = V_{\tau_m}^i \geq \max\left\{L, \bar{V}_{\tau_m}^i\right\}$ so that rolling over is indeed the decision that maximizes the joint surplus. Second, it must be that $B_{\tau_m}^i(c) \geq L$ so that the bank prefers rolling over the loan and receiving the entire interim cash flow over liquidating the project and receiving L. Otherwise, the bank with control rights over the asset will

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[7]We assume the two parties do not bargain over the face value F, with the underlying microfoundation that F is the maximum pledgeable income of the project.

choose to liquidate the asset.

2.5 Strategies and Equilibrium

The public history $\mathcal{H}_t$ consists of time t and the entrepreneur's and the bank's actions up to t. Specifically, it includes at any time $s \leq t$, whether the entrepreneur borrows from the bank or the market and whether the project is liquidated. In the benchmark model, we assume the event of loan maturing is unobservable, which helps with tractability. This assumption is innocuous to the qualitative aspect of our results, as shown in subsection 4.2. For any public history, the strategy of the market is summarized by the price of market debt D_{τ_m}. Given that the market is competitive, the price of debt at which it breaks even satisfies (5).[8]

The private history h_t consists of the public history $\mathcal{H}_t$, the rollover event, as well as the Poisson event on news arrival and of course the content of news. In the case that the financial constraint $y \leq c$ does not bind, Nash Bargaining allows us to treat the bank and the entrepreneur as one entity and their problem is to choose whether to roll over the loan once it matures in order to maximize the joint surplus. In general, a strategy of the entrepreneur is a stopping time that determines the time to switch to market financing. The strategy of the bank specifies whether to roll over the loan at each rollover date τ_m or to liquidate it, in the case it does not receive the full payment F.

Let V_t^i be the joint value of the entrepreneur and the bank in the lending relationship, which satisfies the following Bellman equation:[9]

$$V_t^u = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}\left[q_0 + (1-q_0)\theta\right]R \right.\right. \tag{10a}$$

$$\left.\left. + \mathbb{1}_{\tau=\tau_\lambda}\left[q_0 V_\tau^g + (1-q_0)V_\tau^b\right] + \mathbb{1}_{\tau=\tau_m}\max_{B_\tau^u \geq L}\left\{V_\tau^u, L, \bar{V}_\tau^u\right\} \right]\right\}$$

$$V_t^g = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}R + \mathbb{1}_{\tau=\tau_m}\max_{B_\tau^g \geq L}\left\{V_\tau^g, L, \bar{V}_\tau^g\right\} \right]\right\} \tag{10b}$$

$$V_t^b = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}\theta R + \mathbb{1}_{\tau=\tau_m}\max_{B_\tau^b \geq L}\left\{V_\tau^b, L, \bar{V}_\tau^b\right\} \right]\right\}. \tag{10c}$$

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[8]We offer a micro-foundation as follows. In each period, two short-lived market-based lenders simultaneously enter and make private offers to all entrepreneurs. Those whose loans have matured, i.e., $t = \tau_m$ may choose to accept the offer. This microfoundation will give rise to the *No-Deals* condition as in Daley and Green (2012).

[9]We use the standard notation $\mathbb{E}_{t-}[\cdot] = \mathbb{E}[\cdot|h_{t-}]$, to indicate that the expectations is conditional on the history before the realization of the stopping time τ.

With some abuse of notation, in the first equation $\tau = \min\{\tau_\phi, \tau_\lambda, \tau_m\}$, while in the last two equations $\tau = \min\{\tau_\phi, \tau_m\}$. The first term in all three equations, cds, is the value of interim cash flows over time $[s, s + ds]$. The project matures and pays off the final cash flows if $\tau = \tau_\phi$. If $\tau = \tau_m$, the bank loan matures and the two parties chooses among rolling over, liquidating, or switch to market financing to maximize their value. In the case that the pair is uninformed, news arrives at random time τ_λ, after which they become informed. The maximization at time τ is subject to the additional constraint that the value of the bank at any rollover date has to be greater than L. As before, the indicators variables imply whether the loan is rolled over, the project is liquidated, or the entrepreneur obtains market financing.

Note that we have defined $E_t^i(y)$ and $B_t^i(y)$ as the continuation value of the entrepreneur and the bank at time t, where y is the prevalent loan rate. Sometimes, we will also refer to $E_t^i(y)$ as equity value. By definition, for type $i \in \{g, b\}$

$$E_t^i(y) = \mathbb{E}_{t^-}\left\{\int_t^\tau e^{-r(s-t)}(c-y)ds + e^{-r(\tau-t)}\left[\max\left\{\tilde{R} - F, 0\right\} + \mathbb{1}_{\tau=\tau_m}\left(\mathbb{1}_{\text{rollover}}E_{\tau_m}^i + \mathbb{1}_{\text{market}}\bar{E}_{\tau_m}^i\right)\right]\right\}$$

$$B_t^i(y) = \mathbb{E}_{t^-}\left\{\int_t^\tau e^{-r(s-t)}yds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}\min(\tilde{R}, F) + \mathbb{1}_{\tau=\tau_m}\left(\mathbb{1}_{\text{rollover}}B_{\tau_m}^i\right.\right.\right.$$

$$\left.\left.\left. + \mathbb{1}_{\text{market}}F + \mathbb{1}_{\text{liquidation}}L\right)\right]\right\},$$

where $\bar{E}_{\tau_m}^i = \bar{V}_{\tau_m}^i - F$ is the continuation value of the entrepreneur once she finances with the market, with $\bar{V}_{\tau_m}^i$ defined in (4).

We look for a perfect Bayesian equilibrium of this game.

Definition 1. *An equilibrium of the game satisfies*

1. *Optimality: the rollover decisions are optimal for the bank and the entrepreneur, given the beliefs $\{\pi_t^i, q_t\}$. The rate of the loan at rollover dates solves the Nash bargaining problem (7).*

2. *Belief Consistency: for any history on the equilibrium path, the belief process $\{\pi_t^u, \pi_t^g, \pi_t^b\}$ is consistent with Baye's rule.*

3. *Market Breakeven: the price of the public debt satisfies (5).*

4. *No (unrealized) Deals: for any $t > 0$ and $i \in \{u, g, b\}$, $V_t^i \geq \mathbb{E}\left[\bar{V}_t \big| \mathcal{H}_t\right]$.*

The first three conditions are standard. The *No Deals* condition follows Daley and Green (2012), reflecting the requirement that the market cannot profitably deviate by making an offer that the entrepreneur and the bank will accept.

As standard in the literature, we use a refinement to rule out equilibria that arise only due to unreasonable beliefs off the equilibrium path. Specifically, we restrict the belief on the off-equilibrium to be non-decreasing

Definition 2. *Belief monotonicity is satisfied if q_t – the public's belief that the project is good – is non-decreasing in t. An equilibrium that satisfies belief monotonicity is referred to as a monotonic equilibrium.*

We will that show there is one unique monotone equilibrium. Moreover, we show that if the maturity of the loan is sufficiently long, the equilibrium is unique even without the refinement – that is, that any perfect Bayesian equilibrium is monotone.

2.6   Parametric Assumptions

We make the following parametric assumptions to make the problem non-trivial.

Assumption 1 (Liquidation value)**.**

$$NPV_\delta^b < L < NPV_r^g \tag{11}$$

According to Assumption 1, the NPV of a good project to the bank and the entrepreneur is above its liquidation value, which is in turn above the NPV of a bad project to the bank. Therefore, it is socially optimal to liquidate a bad project but to continue a good project until the maturity date.

Assumption 2 (Risky debt)**.**
$$F > \max\left\{\theta R, L\right\}. \tag{12}$$

Assumption 2 assumes the face value of the debt is above both the liquidation value and the expected repayment; otherwise both the bank loan and the public bond can be safe.

Finally, we assume the size of the interim cash flow c to be higher than rF.

Assumption 3 (interim cash flow)**.**
$$c > rF. \tag{13}$$

2.7 First-best Outcome

Before formally characterizing the equilibrium result, we present the first-best outcome, which is achieved if news could be publicly observable and loans mature instantly. Assumption 1 guarantees that any good project will immediately receive finance from the market, whereas a bad project will be liquidated upon news coming out. Let $NPV_{r\delta}^u$ be the time-0 valuation of the unknown project if it is financed with the bank and switch to market/liquidation upon good/bad news.

$$NPV_{r\delta}^u = \frac{c + \phi \left[q_0 + (1 - q_0)\,\theta\right] R}{r + \phi + \lambda} + \frac{\lambda}{r + \phi + \lambda} \left[q_0 \frac{c + \phi R}{\delta + \phi} + (1 - q_0)\,L\right]. \qquad (14)$$

Proposition 1. *In the first-best outcome, a good project is immediately financed by the market, whereas a bad project is immediately liquidated. If $\{NPV_\delta^u, NPV_{r\delta}^u, L\} = L$, an unknown project will be liquidated. If $\{NPV_\delta^u, NPV_{r\delta}^u, L\} = NPV_\delta^u$, it will be financed by the market. If $\{NPV_\delta^u, NPV_{r\delta}^u, L\} = NPV_{r\delta}^u$, it will be financed by the bank until news comes out.*

3 Equilibrium

In this section, we solve the equilibrium in several steps. In subsection 3.1, we study a benchmark economy by ignoring the financial constraints $y_{\tau_m} \leq c$ and $D_{\tau_m} \geq F$.[10] In subsection 3.2, we describe the equilibrium with a formal treatment of the financial constraints. The equilibrium will be similar to the one in subsection 3.1, except for the boundary conditions. Both subsection 3.1 and 3.2 assume learning as an exogenous process, whereas Subsection 3.3 analyzes the case that learning is a costly decision by banks.

3.1 Benchmark: No financial constraint

The economy is characterized by state variables in private and public beliefs $\{\mu_t^i, \pi_t^i\}$, all of which turn out to be deterministic functions of time elapsed. Therefore, we use t as the state variable. Specifically, the equilibrium can be characterized by two thresholds $\{t_b, t_g\}$, as illustrated by Figure 1. If $t \in [0, t_b]$, the bank and the entrepreneur liquidate the project upon loan maturing if it is already known to bad – *efficient liquidation* region. Loans for

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[10]This case corresponds to an entrepreneur who is less financially constrained but still not deep-pocked enough to finance the entire loan F. In other words, we assume she has enough funds to absorb the rollover losses but not enough to fully repay the bank debt. Alternatively, one can think of this in the traditional trade-off framework with a deep-pocked entrepreneur who uses debt to take advantage of the tax shields. We present one such model in Appendix A.4.

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other types (good and unknown) will be rolled over. If $t \in [t_b, t_g]$, all types of loans will be rolled over, including the bad ones – *extend and pretend* region. Finally, if $t \in [t_g, \infty)$, the two entities will always refinance with the market upon loan maturity – *market financing*.
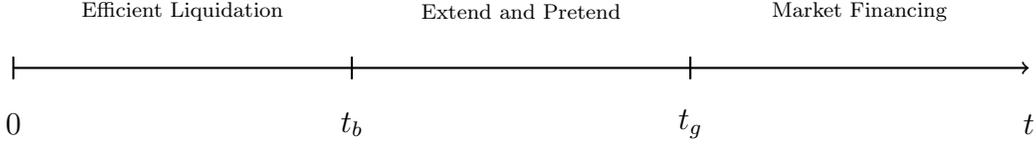

Figure 1: Equilibrium regions

Given the equilibrium conjecture, the evolution of beliefs follow Lemma 1.

Lemma 1. *In a monotone equilibrium with threshold $\{t_b, t_g\}$, beliefs evolve as follows.*

1. *Without liquidation, the public beliefs $(\pi_t^u, \pi_t^g, \pi_t^b)$ satisfy the following differential equation:*

$$\dot{\pi}_t^u = -\lambda \pi_t^u + \mathbb{1}_{t \le t_b} m \pi_t^u \pi_t^b \tag{15a}$$

$$\dot{\pi}_t^g = \lambda \pi_t^u q_0 + \mathbb{1}_{t \le t_b} m \pi_t^g \pi_t^b \tag{15b}$$

$$\dot{\pi}_t^b = \lambda \pi_t^u (1 - q_0) - \mathbb{1}_{t \le t_b} m \pi_t^b (1 - \pi_t^b). \tag{15c}$$

2. *With liquidation, π_t^b jumps to 1, whereas π_t^u and π_t^g jump to 0. Initially, $\pi_0^u = 1$ and $\pi_0^g = \pi_0^b = 0$.*

3. *For any $t < t_b$,*

$$q_t = \frac{q_0 \left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda} - 1} e^{mt}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m - \lambda)s} ds}. \tag{16}$$

4. *For $t > t_b$,*

$$q_t = \bar{q} = \frac{1}{(1 - \theta)} \left(\frac{\delta + \phi}{r + \phi} \frac{c + \phi F}{\phi F} - \frac{c}{\phi F} - \theta\right). \tag{17}$$

According to Lemma 1, the market beliefs depend crucially on type-b's rollover decisions. Figure 2 provides a graphical illustration to the public belief systems for $t < t_g$. The top panel shows π_t^u, which decreases monotonically due to the arrival of news over time. In contrast, π_t^g keeps increasing, since the informed good type gets discovered over time and keeps rolling over the loan. Finally, π_t^b evolves non-monotonically. During $[0, t_b]$, it increases initially as bad types get revealed (note that they don't exit immediately due to the finite maturity of the loan). Ultimately, it starts to decline as more and more of the informed bad

types get liquidated and exit funding. After t passes t_b, becomes no bad type will further liquidate their projects, π_t^b starts to increase again.
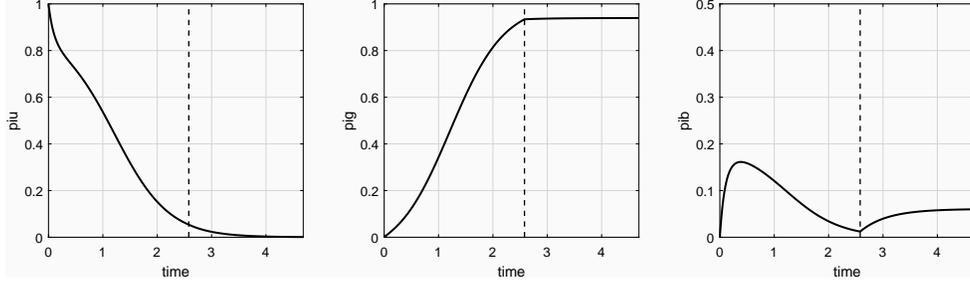

Figure 2: Public Beliefs when $t < t_g$

This figure plots the public beliefs process with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 2$, $q_0 = 0.1$, $\beta = 0.5$.

The evolution of q_t is also straightforward. During $[0, t_b)$, q_t drifts up because bad entrepreneurs learn from news over time and exit. After t goes above t_b, the average quality of borrowers remains unchanged from the market's perspective, that is

$$\dot{q}_t = \dot{\pi}_t^g + q_0 \dot{\pi}_t^u = 0.$$

For the remainder of this subsection, we solve the equilibrium in two steps. In the first step (3.1.1), we treat the bank and the entrepreneur as one entity and solve for t_b and t_g, the optimal choice of timing when they liquidate the project and when they switch to market finance.

3.1.1 Liquidation, Rollover, and Market Financing

By considering the changes in valuation V_t^i, $i \in \{u, g, b\}$ over a small interval $[t, t + dt]$, we are able to derive the following Hamilton-Jacobi-Bellman (HJB) equation system:

$$(r + \phi) V_t^u = \dot{V}_t^u + c + \phi \left[q_0 + (1 - q_0) \theta \right] R \tag{18a}$$
$$+ \lambda \left[q_0 V_t^g + (1 - q_0) V_t^b - V_t^u \right] + m \mathcal{R}(V_t^u, \bar{V}_t^u)$$
$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R + m \mathcal{R}(V_t^g, \bar{V}_t^g) \tag{18b}$$
$$(r + \phi) V_t^b = \dot{V}_t^b + c + \phi \theta R + m \mathcal{R}(V_t^b, \bar{V}_t^b), \tag{18c}$$

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where

$$\mathcal{R}(V_t^i, \bar{V}_t^i) \equiv \max \left\{ 0, \bar{V}_t^i - V_t^i, L - V_t^i \right\} \tag{19}$$

The first term on the right hand side $\dot{V}_t^u$ is the change in valuation which corresponds to the the capital gain; the second term captures the benefits of interim cash flow, and the third term corresponds to the project maturity, with arrival rate ϕ. In the latter case, the bank and the entrepreneur receive a payoff of R with probability $q_0 + (1 - q_0)\,\theta$. The fourth term corresponds to the arrival of news which happens at an arrival rate λ. Following the news, the bank and the entrepreneur become informed about the project. Finally, upon loan maturity which happens with an arrival rate m, the bank and the entrepreneur choose between rolling over the debt (0 in Equation (19)), replacing the loan with the market bond ($\bar{V}_t^i - V_t^i$ in (19)), and liquidating the project ($L - V_t^i$ in (19)). Note that we have assumed a project will be liquidated if $L - V_t^i = \arg\max \mathcal{R}\left(V_t^i, \bar{V}_t^i\right)$, which will no longer be the case with explicit account for financial constraint. Equation (18b) and (18c) can be interpreted in the similar vein.

The three equilibrium regions will differ in $\mathcal{R}\left(V_t^i, \bar{V}_t^i\right)$, i.e., decision when the loan matures. To better explain the economic intuition, we describe the equilibrium backwards in the time elapsed.

Market Financing: $[t_g, \infty)$ In this region, $\mathcal{R}(V_t^i, \bar{V}_t^i)$ is maximized by letting it equals $\bar{V}_t^i - V_t^i$. Also, $\dot{V}_t^i$, $i \in \{u, g, b\}$ is dropped in equations (18a)-(18c) because the belief q_t stays unchanged.

All types will replace their loans with public debt due to the lower cost of market financing. Ultimately, market financing is cheaper because market lenders have lower discount rates, $\delta < r$, and they are willing to refinance with loan at a price that reflects the average quality of the project which exceeds the initial quality q_0. Notice that the average quality of the project is higher than q_0 because in equilibrium, some bad types would have liquidated the project.

Extend and Pretend: $[t_b, t_g)$ Working backward, we now consider the region $[t_b, t_g)$ during which all loans, included bad ones, are rolled over. When time is close to t_g, the bank finds optimal to wait until t_g, so the entrepreneur can replace the matured loan with public bond. Mathematically, on the right-hand-side of equations (18a)-(18c), $\mathcal{R}(V_t^i, \bar{V}_t^i)$ is maximized by letting it equals 0.

During $[t_b, t_g)$, any entrepreneur who seeks financing from the market will be regarded

as bad for sure. Therefore, such a deviation will only show up off the equilibrium path. Equilibrium in this region is clearly inefficient. A bad project should be liquidated but instead, the bank and the entrepreneur roll it over in the hope of sharing the losses with the market lenders after t_g. By not liquidating between 0 and t_b, they have accumulated "good" reputation and as a result, "extend and pretend" can show up in equilibrium.

Efficient Liquidation: $[0, t_b)$ Finally, we focus on the initial region $[0, t_b)$, where bad loans are not rolled over but instead liquidated. At the beginning of the lending relationship, only the uninformed and informed-good types rollover maturing loans. For them, the joint continuation payoff fall below the value at $t = 0$ because no bad news has arrived. By contrast, banks who learn that the project is one of low quality optimally decide to liquidate the projects. Assumption 1 guarantees that liquidation possesses a higher value than continuing the project until the final date t_ϕ. By continuity, liquidation still has a higher payoff if type b needs to wait for a long time (until t_g) to refinance. Mathematically, on the right-hand-side of equations (18a)-(18c), $\mathcal{R}(V_t^b, \bar{V}_t^b)$ is maximized by letting it equals $L - V_t^b$, whereas $\mathcal{R}(V_t^g, \bar{V}_t^g)$ and $\mathcal{R}(V_t^u, \bar{V}_t^u)$ are still maximized by letting it equals 0. The equilibrium is socially efficient in this region.

Boundary Conditions: The following two boundary conditions are needed to pin down $\{t_b, t_g\}$

$$V_{t_b}^b = L \tag{20a}$$

$$\dot{V}_{t_g}^g = 0. \tag{20b}$$

(20a) is the indifference condition for the bad type to liquidate at t_b. This is the traditional value matching condition in optimal stopping problem. In this case, rolling over brings exactly the same payoff L and thus by continuity and monotonicity, she prefers liquidating when $t_m < t_b$ and rolling over when $t_m > t_b$. The second condition, smooth pasting, comes from the *No-Deals* condition. We show in Lemma 4 of Appendix A.1.2 that if this conditions fails then the type g will have strictly higher incentives to switch to market financing before t_g, which constitutes an arbitrage opportunity for market participants. Essentially, the *No-Deals* guarantees the equilibrium will ultimately be one with pooling, and given so, the smooth-pasting condition solves the optimal-stopping time problem for the good types. The smooth-pasting condition picks the earliest t_g for the entrepreneur to switch to market financing. Given the boundary conditions, we can uniquely pin-down $\{t_b, t_g\}$, which is given by the following proposition.

Proposition 2. *In absence of financial constraints, there is a a unique monotone equilibrium characterized by rollover thresholds t_b and t_g given by*

$$t_g = t_b + \frac{1}{r+\phi} \log \left(\frac{(r+\phi)V_{t_g}^b - (c+\phi\theta R)}{(r+\phi)L - (c+\phi\theta R)} \right). \tag{21}$$

and

$$t_b = \min\{t : q_t = \bar{q}\}, \tag{22}$$

where

$$V_{t_g}^b = \frac{c+\phi R}{r+\phi} - \frac{\phi R (1-\theta) + m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r+\phi+m}. \tag{23}$$

Figure 3 plots the value function of all three types. In this example, the equilibrium $t_b = 5.48$ and $t_g = 8.22$. In all three panels, the blue solid lines stand for the value function, whereas the red dashed line shows the levels of L. Clearly, all three value functions stay constant after t passes t_g. In fact, as shown in Lemma 5 of Appendix A.1.2, V_g stays a constant throughout the entire range. In other words, the informed-good types always expect the same continuation value. By contrast, the value of informed-bad types (bottom panel) exceeds L only after t passes t_b and then increases sharply until $t = t_g$.
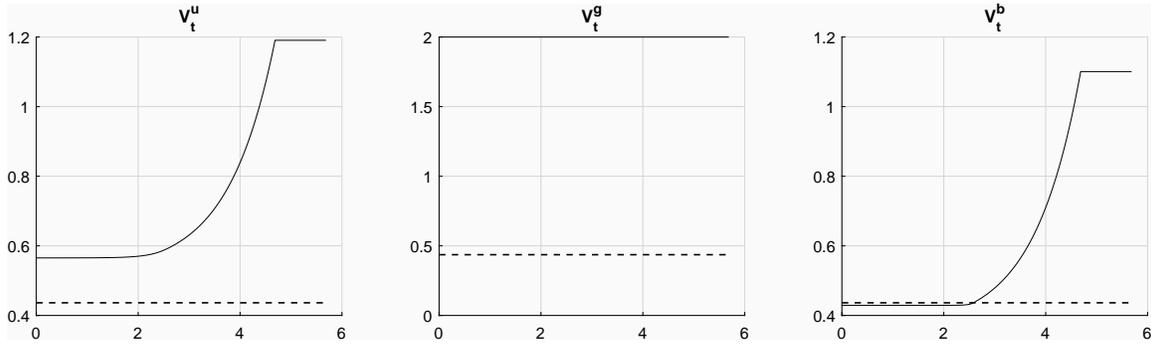

Figure 3: Value Functions

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 2$, $q_0 = 0.1$, $\beta = 0.5$.

Finally, we discuss the equilibrium uniqueness without the monotone belief refinement.

Proposition 3. *There exists a unique pair $\{\underline{m}, \bar{m}\}$ satisfying $\underline{m} < \bar{m}$ such that the equilibrium described in Proposition 2 is unique if and only if $m \in (\underline{m}, \bar{m})$. If $m > \bar{m}$, the equilibrium is not unique. If $m < \underline{m}$, the unique equilibrium is one in which $t_b = t_g$.*

Remark. The results will stay unchanged if we allow the entrepreneur to renegotiate and prepay the bank loans. During $[0, t_g)$, renegotiation is never triggered. After t_g, all bank loans will immediately be renegotiated. As a result, for all three types, $V_{t_g}^i = \bar{V}^i$. According to Proposition 2, t_b stays unchanged, whereas t_g gets even higher.

3.1.2 Entrepreneur and Bank Value

In the second step, we study how the joint surplus of the entrepreneur and the bank is distributed between the two parties. We will mainly describe the results graphically and leave the analytical details to Appendix A.2, including the HJB equations.
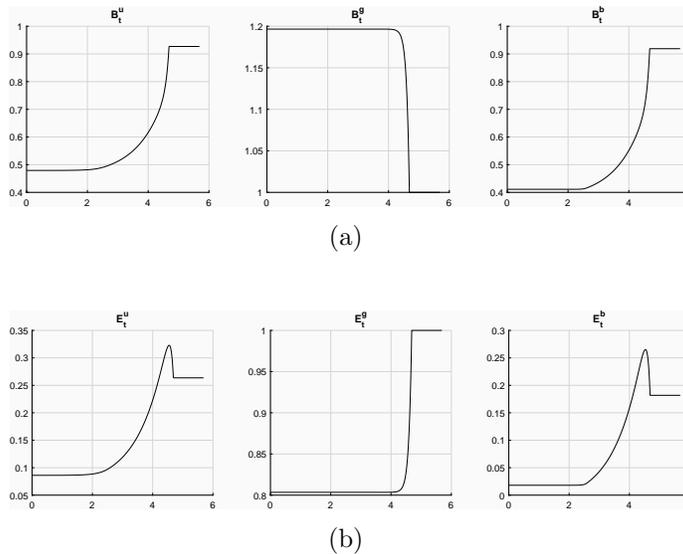

(a)

(b)

Figure 4: Bank and Entrepreneur Value Functions

This figure plots the bank's (a) and entrepreneur's (b) value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV^b$, $\lambda = 2$, $q_0 = 0.1$, $\beta = 0.5$.

Figure 4 plot the value functions. A prominent feature of is the non-monotonicity of the entrepreneurs' value function E_t^u and E_t^b. Intuitively, there are two forces at work here. First, the value of a bad project is increasing, as time gets closer and closer to the market financing stage. As a result, the surplus of rolling over the bad loan $V_t^b - L$ gets larger. Ceteris paribus, the entrepreneur's value function should increase. However, there is a second, counterveiling force. During the market financing stage, the disagreement point in the Nash bargaining game is $(0, L)$: if the bargaining does not reach an agreement, the bank only receives the liquidation value L. During the market financing stage, however, the bank will always gets fully repaid and thus receives F. As time t gets closer to the market

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financing stage, the entrepreneur's ability to "hold up" the bank gets more limited because it is increasingly likely that the next roll-over event will occur during the market financing stage. Therefore, the entrepreneur's value function decreases. Given the opposite effects of these two forces, the overall effect can be non-monotonic.[11] In subsection 4.1, we show that in the case with instantly-maturing debt, the first force will lead to the entrepreneur's value function to increase until t reaches t_g^-. At t_g, it experiences a discontinuous downwards jump due to the second effect.

3.2 Binding Financial Constraint

By relaxing financial constraints, we have assumed all rollover decisions were made to maximize the joint surplus of the bank and the entrepreneur. Specifically, at each rollover date, a loan would be rolled over if the joint surplus is above the liquidation value L. In this subsection, we formally analyze the model with financial constraint. As a result, at rollover dates between t_b and t_g, the newly negotiated loan rate y cannot exceed the rate of interim cash flow c. This constraint limits the size of the transfer from the entrepreneur to the bank. Moreover, at rollover dates after t_g, the price of the market debt D_{τ_m} must be sufficient to cover the face value of the loan F. As we will see shortly, the bank will sometimes liquidate the project and get L even though the joint surplus is above L.

The HJBs for the value function $\{V_t^i, \ i \in \{u, g, b\}\}$ are the same as those in subsection 3.1.1. Again, we can use two thresholds $\{t_b, t_g\}$ to characterize the equilibrium solutions. One may wonder whether the financial constraint could be always slack. Lemma 2 shows this is never possible

Lemma 2. *The financial constraint $y \leq c$ always bind at t_b.*

We offer a heuristic proof as follows. Suppose by contradiction that the constraint is always slack, the boundary condition at t_b, characterized by (20a), immediately leads to $B_{t_b}^b(y_{t_b}) = L + (1 - \beta)(V_{t_b}^b - L) = L$. This implies that if a bad loan matures at t_b, the bank will receive a continuation value L, and as a result, the bad entrepreneur's continuation value is 0. This constitutes a violation, as the bad entrepreneur could always wait until t_g and finance with the market, which guarantees a strictly positive payoff.

3.2.1 Equilibrium Boundaries

Let us now turn to the the boundary conditions under financial constraints. First, the smooth-pasting condition $\dot{V}_{t_g}^g = 0$ continues to hold. Intuitively, this condition follows from

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[11]As we show in Lemma 8 in the Appendix, the sign of E_t^b can only change sign at most once though.

the *No Deals* condition, which essentially selects an equilibrium in which a good-type entrepreneur chooses to refinance with the market as early as possible. Note that the smooth-pasting condition pins down $\bar{q}$, the average quality of entrepreneurs during $[t_b, t_g)$. Moreover, the financial constraint also requires $D_{\tau_m} \geq F$ so that the entrepreneur can raise sufficient funds from the market to repay the bank. (6) implies this requires $\bar{q} \geq q_{\min}$, which naturally follows if Assumption 3 holds. Therefore, the average quality of entrepreneur after t_b is identical to the case without financial constraint, and $t_b = \min\{t : q_t \geq \bar{q}\}$ also remains unchanged.

The second boundary condition, value matching condition will be different. In particular, since the entrepreneur is financially constrained and cannot repay its loan, it is the bank that decides whether to liquidate the project. Therefore, the value-matching condition at t_b becomes

$$B_{t_b}^b(c) = L. \tag{24}$$

Note that we have used the result from Lemma 2 that the constraint always binds for type b at $t = t_b$. Depending on parameters, the constraint may or may not bind at $t = t_g$. Proposition 4 summarizes the outcome that it always binds. The other case is shown in the proof in the appendix.

Proposition 4. *If*

$$L - \frac{c + (\phi\theta + m)F}{r + \phi + m} + (1 - \beta)\left[\frac{c}{r + \phi} + \frac{\phi}{r + \phi}\left((1 - \theta)\frac{mF}{r + \phi + m} + \theta R\right) - L\right] > 0,$$

then the equilibrium characterized by two thresholds $\{t_b, t_g\}$, where $\{\bar{q}, t_b\}$ are identical to those in Proposition 2 and t_g is given by

$$t_g = t_b + \frac{1}{r + \phi} \log\left(\frac{(r + \phi)L - (c + \phi\theta F)}{\frac{r + \phi}{r + \phi + m}(c + \phi\theta F + mF) - (c + \phi\theta F)}\right). \tag{25}$$

How does the financial constraint affect t_g and the length of the zombie lending period $t_g - t_b$? Since $E_{t_b}^b(c) > 0$, (24) implies at $t = t_b$, the joint surplus $V_{t_b}^b > L$. Therefore,

Proposition 5. *The length of the zombie period $t_g - t_b$ gets shorter under the financial constraint.*

Intuitively, the incentives for bad types to mimic others are mitigated under the financial constraint, since there is a limit to the size of the transfer that a bad entrepreneur can use to "bribe" the bank from liquidation. The presence of financial constraints actually mitigates the lemon's problem.

3.2.2 Initial Decisions

Now that we have described for the equilibrium with financial constraint, we will solve for the maximum borrowing amount that an entrepreneur can raise from a bank at time 0. We will also solve for the amount that she can borrow from the market. A comparison will illustrate what types of firms choose bank versus market financing with different levels of initial conditions.

3.3 Endogenous Learning

Our analysis has so far assumed learning as an exponential process as long as the entrepreneur borrows from the bank. In this subsection, we consider the situation in which learning is endogenously chosen by the bank as a costly decision. In order to simplify the analysis, we restrict to the case in which the entrepreneur is financially unconstrained. Given a learning rate chosen by the bank $a_t \in [0,1]$, the news process arrives at intensity λa_t. Moreover, the cost of learning is linear in a_t and given by ψa_t. Let us continue to study an equilibrium with rollover thresholds $\{t_b, t_g\}$ that is described in subsection 3.1.1. As we are assuming that the entrepreneur is unconstrained, at any rollover date $\tau_m \in [t_b, t_g]$, $B^i_{\tau_m^+} = O^i_{\tau_m} + (1 - \beta)\left(V^i_{\tau_m} - O^i_{\tau_m}\right)$.

At any time $t \in (0, t_g)$, the bank's continuation value satisfies the HJB equation

$$(r + \phi + m)\, B^u_t = rF + \phi[q_0 + (1 - q_0)\theta]F + \dot{B}^u_t$$
$$+ \max_{a \in [0,1]} \left\{ \left(\lambda(q_0 B^g_t + (1 - q_0)B^b_t - B^b_t) - \psi\right)a \right\} + m[L + (1 - \beta)(V^u_t - L)].$$

Clearly, $a_t \geq 0$ if and only if

$$q_0 B^g_t + (1 - q_0)B^b_t - B^b_t \geq \frac{\psi}{\lambda}$$

.

In any equilibrium that involves the three regions identified in subsection 3.1.1 and two thresholds $\{t_b, t_g\}$, we show in the appendix that the bank's learning policy is always bang-bang: $a_t = \mathbf{1}_{\{t < t_a\}}$ for some threshold t_a to be determined. Moreover, it must be the case that $t_a \leq t_b$. Intuitively, without liquidating, the value of an uninformed bank is a linear combination between an informed-good and an informed-bad. With liquidating, however, the value of an informed-bad is protected by L so the payoff is convex in the type (see Figure 5 for a graphical illustration.). In this case, information is valuable and learning will be endogenously chosen if the cost is small enough.
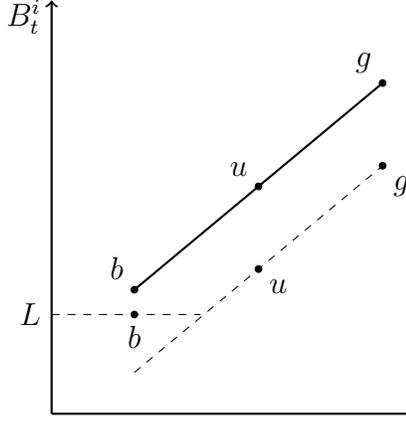
Figure 5: Graphical Illustration of Learning Benefits

Under the constructed equilibrium, beliefs on $(0, t_a)$ are still given by (15a) - (15c) while on (t_a, t_b), beliefs evolve as

$$\dot{\pi}_t^u = m\pi_t^u \pi_t^b$$
$$\dot{\pi}_t^g = m\pi_t^g \pi_t^b$$
$$\dot{\pi}_t^b = -m\pi_t^b \left(1 - \pi_t^b\right).$$

Proposition 6 summarizes the results.

Proposition 6. *If*

$$\frac{\psi}{\lambda} < \frac{m(1-\beta)(1-q_0)}{r + \phi + m} \left(L - \frac{c + \phi\theta R}{r + \phi}\right),$$

then there is an equilibrium characterized by three thresholds $\{t_a, t_b, t_g\}$, *where* $t_a < t_b < t_g$ *and the monitoring threshold satisfies*

$$t_a > \frac{1}{\lambda} \log \left(\frac{\bar{q}}{1 - \bar{q}} \frac{1 - q_0}{q_0}\right).$$

The rollover decision is identical to the one in Proposition 2. The bank monitoring strategy is $a_t = \mathbb{1}_{t < t_a}$ *so* $\lambda_t = \mathbb{1}_{t < t_a}\lambda$.

4 Extensions and Robustness

4.1 Instantly-Maturing Debt

In this subsection, we show the solution to a special case of our model – instantly-maturing debt. Specifically, we take the maturity intensity of the debt m to infinity and

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study how the solution depends on primitives. We will state the main results, with details supplemented in Appendix A.3.

Proposition 7. *When bank loans mature instantly,*

$$q_t = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t}} \quad \forall t < t_b \tag{26a}$$

$$t_b = \frac{1}{\lambda} \left[\log\left(\frac{\bar{q}}{1 - \bar{q}}\right) - \log\left(\frac{q_0}{1 - q_0}\right) \right] \tag{26b}$$

$$t_g = t_b + \frac{1}{r + \phi} \log\left(\frac{\phi(1 - \theta)F}{(r + \phi)L - c - \phi\theta R}\right) \tag{26c}$$

$$\bar{q} = \frac{1}{(1 - \theta)} \left\{ \frac{1}{\phi} \left[\frac{(\delta + \phi)^2}{r + \phi} - \delta \right] - \theta \right\}. \tag{26d}$$

When bank loans mature instantly, a project is immediately liquidated once it is known as bad before t_b. In this case, the length of this efficient liquidation region only depends on the speed of learning λ and the ultimate credit quality $\bar{q}$.

Entrepreneur's value function

Next, we study the valuation of entrepreneur before t reaches t_g. We will focus on the informed-bad case and leave the other cases in the appendix. With instant maturing loans, the contract is renegotiated continuously. Therefore, for $t < t_b$, it is immediately clear that $E_t^b = 0$ and $B_t^b = L$. For $t \in (t_b, t_g)$, let $G_t^i dt$ be the rollover gains that entrepreneur receive during $[t, t + dt]$. In this case, the HJB of entrepreneur becomes

$$(r + \phi) E_t^b = \dot{E}_t^b + \phi\theta (R - F) + (c - r)F + G_t^b. \tag{27}$$

Lemma 3. *For the instantly-maturing debt, when $t \in (t_b, t_g)$, the entrepreneur receives rollover gains $G_t^i dt$ where*

$$G_t^b = \beta \left[(r + \phi\theta)F - (r + \phi) L \right] - (1 - \beta) \left[\phi\theta (R - F) + (c - r)F \right] \tag{28}$$

The rollover gains of the other two types, as well as the proof for Proposition 1 are in the Appendix. Clearly, $G_t^b > 0$ for large enough β. When the entrepreneur has more bargaining power, she receives gains by constantly renegotiating the loan contract.

The non-monotonicity in entrepreneur's value function can be easily seen in the case of instantly-maturing debt. Lemma 3 implies $\dot{E}_t^b = (r + \phi) \left(E_t^b + \beta L \right) - \beta \left[\phi\theta R + c \right]$. Also

according to Lemma 8 implies in the Appendix, E_t^b is non-monotonic on $[t_b, t_g]$ if and only if $\dot{E}_{t \to t_g^-}^b < 0$, which will be the case if the liquidation value is low enough.

Proposition 8. *In the case of instantly-maturing debt, E_t^b as a downward jump at t_g if and only if*

$$L < F - \frac{1-\beta}{\beta} \frac{\phi\theta(R-F)+(c-r)F}{r+\phi}. \tag{29}$$

Note that (29) holds when $\beta = 0$ but can never hold when $\beta = 1$. That is, the non-monotonic pattern is more prominent when the entrepreneur has more bargaining power. Intuitively, the non-monotonicity happens because the outside option of the bargaining experience a discontinuous jump at $t = t_g$. Prior to that, the bank can liquidate the project for a value of L, whereas the entrepreneur receives nothing. For low levels of L, such non-monotonicity pattern is more prominent. When the entrepreneur has more bargaining power, she can extract more of the surplus early on before t reaches t_g. If her bargaining power gets high enough, the entrepreneur is essentially preventing the bank from extracting a large fraction of the surplus. This extraction can improve the entrepreneur's value so much that it even exceeds the future value after financing with the market.

4.2 Observable Rollover

So far, we have assumed rolling over a loan is not observed by the market participants. In practice, however, the maturity event is sometimes observable. Next, we maintain most of the assumptions in the benchmark model but introduce two minor modifications to simplify the analysis. First, whenever a loan matures it is observable if the bank decides to rollover or liquidate. Second, the maturity of loans is publicly known to be fixed at $1/m$ as opposed to follow an exponential distribution. The deterministic maturity is simpler in this case as we can work backward over the sequence of rollover times. Notice that the model is essentially identical to one written in discrete time. Thus, as in most discrete times of asymmetric information with binary types, the equilibrium strategy will in general involve mixed strategies due to an integer problem. For the remainder of this subsection, we will construct an equilibrium which has features to those described in Proposition 2. We will also show that as $m \to \infty$ so that loans are instantly maturing, this equilibrium converges to the one we have seen in the benchmark model.

Let $n \in \{1, 2, 3 \cdots\}$ be the sequence of rollover events. The date associated to the n-th rollover event is $t_n = n/m$ and the time between two rollover dates is $1/m$. As before, we can construct an equilibrium with two thresholds: t_b and t_g. However, with deterministic rollovers it is notationally more convenient to specify the two thresholds in term of the rollover events: n_b, n_g. The following proposition describe the equilibrium.

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Proposition 9. *If loan maturity is fixed at $1/m$ and rollover is observable, then there exists $\{n_b, n_g\}$ such that*

1. *Efficient liquidation*

   (a) *For $n < n_b$, all bad projects are liquidated, whereas other projects are rolled over;*

2. *Zombie lending*

   (a) *For $n = n_b$, a fraction $\alpha \leq 1$ of the bad projects are liquidated, whereas other projects are rolled over.*

   (b) *When $n \in (n_b, n_g - 1)$, all projects are rolled over.*

3. *Market Financing*

   (a) *When $n = n_g$, market lenders make an offer at $\bar{V}$ with probability $\beta \leq 1$.*

   (b) *When $n = n_g + 1$, all loans are sold to the market.*

Under observable rollovers, the public belief for the loan quality q_t stays unchanged for any time between the two roll-over events. At the roll-over event $n < n_b$ or equivalently $t < n_b/m$, the belief will experience a discrete jump. If the project is liquidated, clearly q_t jumps to 0. If the loan is rolled over, q_t actually follows the process described in the instantly-maturing debt, as shown in (26a). Note there is an equivalence in beliefs under fixed maturity and instantly-maturing exponential debt, because the event of maturing is occurring with certainty at $t = n/m$. q_t stays unchanged at $\bar{q}$ after $t > n_b/m$, implies that the continuation value by selling to the market also stays at $\bar{V}$. With some abuse of notation, let V_n^i be the continuation value at the n-th rollover time (i.e. at time n/m. In that case, we have

$$V_{n_g-1}^g = \int_0^{\frac{1}{m}} e^{-(r+\phi)s} \left(c + \phi R\right) ds + e^{-\frac{r+\phi}{m}} V_{n_g}^g.$$

To simplify notation, let

$$\nu^g \equiv \int_0^{\frac{1}{m}} e^{-(r+\phi)s} \left(c + \phi R\right) ds$$

be the flow payoff between two rollover events, which is time independent. We can rewrite $V_{n_g-1}^g = \nu^g + e^{-\frac{r}{m}} V_{n_g}^g$ which has to be greater or equal than $\bar{V}$. On the other hand, the *No Deals* condition also requires $V_{n_g-1}^g \geq \bar{V}$. Combining these two conditions we find that

$$\nu^g + e^{-\frac{r}{m}} V_{n_g}^g = V_{n_g-1}^g \geq \bar{V} \geq \nu^g + e^{-\frac{r}{m}} \bar{V}.$$

This means that $q_{n_g/m} = \bar{q}$. Because q_t is constant after the n_b-th rollover date, it has to be the case that $q_{n_b/m} = \bar{q}$. Let

$$\hat{q}(n) = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda n/m}}$$

be the beliefs after n rolled over period given the market conjecture that a bad loan is not rolled over, and let $\hat{n} \equiv \min\{n : \hat{q}(n) \geq \bar{q}\}$. If $\hat{n}/m$ is an integer, then $n_b = \bar{n}$ and $\alpha = 0$. However, if $\hat{n}/m$ is not an integer then $n_b = \hat{n} - 1$, and fraction α of the bad projects is liquidated so that the belief conditional on a rollover at n_b is $\bar{q}$. In this case, α satisfies

$$\bar{q} = \frac{(1 - \alpha)\left(1 - q_{\frac{n_b-1}{m}}\right)}{q_{\frac{n_b-1}{m}} + (1 - \alpha)\left(1 - q_{\frac{n_b-1}{m}}\right)}. \tag{30}$$

The definition of $\hat{n}$ together with equation (30) uniquely determine n_b and α. It is only left to determine n_g and β. We do this by turning our attention to the bad type incentive compatibility constraint. The bad type has to be indifferent between liquidating and continue rolling over after n_b. Let $\hat{V}(n', n_b)$ be the payoff if the bad type rolls over until period n' and receives a payoff $\bar{V}^b$, which is given by:

$$\hat{V}^b(n', n_b) \equiv \int_0^{\frac{n'-n_b}{m}} e^{-(r+\phi)s}\left(c + \phi R\right)ds + e^{-\frac{n'-n_b}{m}}\bar{V}^b$$

For fixed n_b, the function $\hat{V}(n', n_b)$ is increasing in n'. Let's define $\tilde{n} \equiv \max\{n' : \hat{V}^b(n', n_b) \geq L\}$. If $\hat{V}^b(\tilde{n}, n_b) = L$, then we can set $n_g = \tilde{n}$ and $\beta = 1$. Otherwise, we have that $\hat{V}^b(\tilde{n}, n_b) > L$ and $\hat{V}^b(\tilde{n}+1, n_b) < L$ so a mixed strategy is required. In particular, if we set $n_g = \tilde{n}$ and choose β such that

$$\beta\hat{V}^b(n_g, n_b) + (1 - \beta)\hat{V}^b(n_g + 1, n_b) = L,$$

we get that $V_{n_b}^b = L$ so the low type is indifferent between liquidating and rolling over. Finally, because the market investors make zero profit, they are willing to mix between the two debt prices at the rollover period n_g. Moreover, we have the following corollary

Corollary 1. *If $m \to \infty$, the equilibrium converges to the one in Proposition 7.*

5   Empirical Implications

Our paper offers a dynamic theory of relationship lending. A first prediction of the model is banks may have endogenous incentives to roll over bad loans. The results on "extend and pretend" largely remind the popular discussions on how securitization induces agency conflicts. Specifically, as documented by existing studies (Agarwal et al., 2011), mortgage lenders and loan servicers rarely wrote off losses shortly after borrowers got financially distressed.

6   Conclusion

In this paper, we introduce private learning into a banking model and study the dynamic tradeoffs of relationship-based lending. Compared to market financing, bank financing enables learning about the quality of the project being financed, but is also subject to the downside of information monopoly and information-monopoly cost. We construct an equilibrium in which an entrepreneur starts with bank financing and subsequently switch to market financing. We characterize the timing of such a switch and study how it is affected by factors such as debt maturity, project illiquidity, credit rating and learning. Our model generates several novel results: 1) Endogenous zombie lending, i.e. the bank is willing to roll over loans known to be bad for the prospect of future loan sales. 2) Short maturity could encourage zombie lending and deteriorate credit quality; and 3) the information-monopoly cost may increase or decrease with the length of the lending relationship.

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A Appendix

A.1 Proofs

A.1.1 Proof of Lemma 1

Proof. The proof relies on the filtering formula for counting processes in Lipster and Shiryaev (Chapter 19). Let x_t^i be the probability that a type $i \in \{g, b, u\}$ firm looks for external financing at time t and let ℓ_t^i be the probability that a type i firm liquidates at time t. Let L_t be the counting process associated to the liquidation time and and M_t be the counting process associated with going to the market. If we denote the type of the firm at time t by $i(t)$ then L_t has intensity $m\ell_t^{i(t)}$ while M_t has intensity $mx_t^{i(t)}$. The process $i(t)$ has transitions governed by the infinitesimal generator

$$\Lambda \equiv \begin{pmatrix} -\lambda & \lambda q_0 & \lambda(1 - q_0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Theorem 19.6 (and following similar calculations to the ones in Examples 2 and 3 therein) we get that

$$d\pi_t^u = -\lambda \pi_t^u dt + \pi_t^u \left(\frac{(\ell_t^u - \ell_t^b)(1 - \pi_t^u) - (\ell_t^g - \ell_t^b)\pi_t^g}{\pi_t^u \ell_t^u + \pi_t^g \ell_t^g + \pi_t^b \ell_t^b} \right) \cdot [dL_t - m(\pi_t^u \ell_t^u + \pi_t^g \ell_t^g + \pi_t^b \ell_t^b)dt]$$

$$+ \pi_t^u \left(\frac{(x_t^u - x_t^b)(1 - \pi_t^u) - (x_t^g - x_t^b)\pi_t^g}{\pi_t^u x_t^u + \pi_t^g x_t^g + \pi_t^b x_t^b} \right) \cdot [dM_t - m(\pi_t^u x_t^u + \pi_t^g x_t^g + \pi_t^b x_t^b)dt]$$

From here, we get that in absence of liquidation and market financing beliefs are given by

$$\dot{\pi}_t^u = -\lambda \pi_t^u - m\pi_t^u \left((\ell_t^u + x_t^u - \ell_t^b - x_t^b)(1 - \pi_t^u) - (\ell_t^g + x_t^g - \ell_t^b - x_t^b)\pi_t^g \right)$$

Suppose that $\ell_t^b = 1$ and $\ell_t^u = \ell_t^g = x_t^u = x_t^g = x_t^b = 0$, then we have

$$\dot{\pi}_t^u = -\lambda \pi_t^u + m\pi_t^u \pi_t^b$$

Similarly, we get

$$\dot{\pi}_t^g = \lambda q_0 \pi_t^u - m\pi_t^g \left[(\ell_t^g + x_t^g - \ell_t^b - x_t^b)(1 - \pi_t^g) - (\ell_t^u + x_t^u - \ell_t^b - x_t^b)\pi_t^u \right]$$

$$\dot{\pi}_t^b = \lambda(1 - q_0)\pi_t^u - m\pi_t^b \left[(\ell_t^b + x_t^b - \ell_t^g - x_t^g)(1 - \pi_t^b) - (\ell_t^u + x_t^u - \ell_t^g - x_t^g)\pi_t^u \right]$$

so in the particular case that $\ell_t^b = 1$ and $\ell_t^u = \ell_t^g = x_t^u = x_t^g = x_t^b = 0$, then we get

$$\dot{\pi}_t^g = \lambda q_0 \pi_t^u + m \pi_t^g \pi_t^b$$
$$\dot{\pi}_t^b = \lambda (1 - q_0) \pi_t^u - m \pi_t^b (1 - \pi_t^b)$$

$\square$

A.1.2 Value function and boundary condition

Lemma 4. *The No Deals condition implies the good type's value function must satisfy smooth-pasting at $t = t_g$. That is*

$$\dot{V}_{t_g}^g = 0.$$

Proof. We prove by contradiction. Suppose $\dot{V}_{t_g-}^g < 0$, then Equation (**??**) implies $V_t^g < \frac{c+\phi R}{(r+\phi)}$. However, this is impossible because $\frac{c+\phi R}{(r+\phi)}$ is the continuation value of the good types if they never finance with the market.

Next, let us assume $\dot{V}_{t_g-}^g > 0$. Under the constructed equilibrium, $\dot{q}_t = 0$ for any $t > t_b$. As a result, $\bar{V}_t^g$ – the continuation payoff when the good type financed with the market at time t also stays at a constant after t_b. Let is be $\bar{V}^g$. If $\dot{V}_{t_g-}^g > 0$, that implies that for ε sufficiently small, $V_{t_g-\varepsilon}^g < \bar{V}^g$ so that the *No Deals condition* fails. Note that this step relies on the fact that $\bar{V}_t^g$ stays a constant for $t \in [t_b, t_g]$. In the equilibrium without the zombie lending stage ($m < m^*$), this condition no longer holds so that in general, $\dot{V}_{t_g}^g \geq 0$.

$\square$

Lemma 5. V_t^g *stays at a constant in any equilibrium that is constructed under t_b and t_g.*[12]

Proof. This directly follows after plugging (**??**) into (**??**) and (**??**). $\square$

A.1.3 Proof of Proposition 2

Proof. By applying the smooth pasting condition

$$V_{t_g}^g = \frac{c + \phi R}{r + \phi} = \frac{c + \phi R + m\bar{V}^g}{r + \phi + m},$$

we get

$$\bar{q} = \frac{1}{(1-\theta)} \left(\frac{\delta + \phi}{r + \phi} \frac{c + \phi F}{\phi F} - \frac{c}{\phi F} - \theta \right)$$

---

[12]This is true under any equilibrium that we construct, which consists of thresholds $\{t_b, t_g\}$. However, it may not hold under any arbitrary equilibrium, which could exist when m gets very large.

after some derivations. Clearly, the equation system in the last region shows

$$V_{t_g}^g - V_{t_g}^b = \frac{\phi R\left(1 - \theta\right) + m\left(\bar{V}^g - \bar{V}^b\right)}{r + \phi + m} = \frac{\phi R\left(1 - \theta\right) + m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r + \phi + m}.$$

In that case, using the same smooth pasting condition, we get

$$V_{t_g}^b = \frac{c + \phi R}{r + \phi} - \frac{\phi R\left(1 - \theta\right) + m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r + \phi + m}.$$

Given that, let us solve for $t_g - t_b$ using the ODE system in region 2. In particular, for any $t \in [t_b, t_g]$,

$$V_t^b = e^{(r+\phi)(t-t_g)}V_{t_g}^b + \frac{c + \phi\theta R}{r + \phi}\left[1 - e^{(r+\phi)(t-t_g)}\right].$$

Using the boundary condition $V_{t_b}^b = L$, we can get

$$t_g - t_b = -\frac{1}{(r + \phi)}\log\left(\frac{L - \frac{c+\phi\theta R}{r+\phi}}{V_{t_g}^b - \frac{c+\phi\theta R}{r+\phi}}\right).$$

The threshold t_b is determine by the condition

$$t_b = \min\{t : q_t = \bar{q}\}.$$

The final step is to find the solution for q_t in the interval $[0, t_b]$. The ODE system in region 1 is

$$\begin{aligned}
\dot{\pi}_t^u &= -\lambda\pi_t^u + m\pi_t^u\pi_t^b \\
\dot{\pi}_t^g &= \lambda\pi_t^u q_0 + m\pi_t^g\pi_t^b \\
\dot{b}_t &= \lambda\pi_t^u\left(1 - q_0\right) - m\pi_t^b\left(1 - \pi_t^b\right).
\end{aligned}$$

Let us define $z_t = \frac{\pi_t^g}{\pi_t^u}$, then,

$$\begin{aligned}
\dot{z}_t &= \frac{\dot{\pi}_t^g\pi_t^u - \pi_t^g\dot{\pi}_t^u}{(\pi_t^u)^2} = \frac{\dot{\pi}_t^g}{\pi_t^u} - z_t\frac{\dot{\pi}_t^u}{\pi_t^u} \\
&= \lambda q_0 + mz_t\left(1 - \pi_t^g - \pi_t^u\right) - z_t\left(-\lambda + m\left(1 - \pi_t^g - \pi_t^u\right)\right) \\
&= \lambda\left(q_0 + z_t\right).
\end{aligned}$$

Therefore, we have the solution

$$z_t = q_0 \left(e^{\lambda t} - 1 \right)$$

$$\Rightarrow$$

$$\pi_t^g = q_0 \left(e^{\lambda t} - 1 \right) \pi_t^u. \tag{31}$$

Since $\pi_t^u + \pi_t^g + \pi_t^b = 1$, we also have

$$\pi_t^b = 1 - \left(q_0 e^{\lambda t} + 1 - q_0 \right) \pi_t^u. \tag{32}$$

Substituting (31) and (32) into the ODE system, we get a first-order ODE for π_t^u

$$\dot{\pi}_t^u = (m - \lambda) \pi_t^u - m \left(q_0 e^{\lambda t} + 1 - q_0 \right) (\pi_t^u)^2,$$

which corresponds to a continuous-time Riccati equation. This equation can be transformed it into a second-order ODE. Let $v_t = -m \left(q_0 e^{\lambda t} + 1 - q_0 \right) \pi_t^u$ and $R_t = q_0 e^{\lambda t} + 1 - q_0$,

$$\dot{v}_t = v_t^2 + \frac{v_t}{R_t} \left[q_0 e^{\lambda t} \lambda + R_t (m - \lambda) \right]. \tag{33}$$

Further, if we let $v_t = -\frac{\dot{y}_t}{y_t} \Rightarrow \dot{v}_t = -\frac{\ddot{y}_t}{y_t} + (v_t)^2$, then we can transforme equation (33) into the following second-order ODE

$$\ddot{y}_t = \frac{\dot{y}_t}{R_t} \left[q_0 e^{\lambda t} \lambda + R_t (m - \lambda) \right]$$

From here, we get that

$$\dot{y}_t = \dot{y}(0) e^{\int_0^t \lambda \frac{1}{1 + \frac{1-q_0}{q_0} e^{-\lambda s}} ds + (m - \lambda) t}$$

Moreover,

$$\int_0^t \frac{1}{1 + \frac{1-q_0}{q_0} e^{-\lambda s}} ds = \frac{1}{\lambda} \log \left(1 - q_0 + q_0 e^{\lambda t} \right)$$

so

$$\dot{y}_t = \dot{y}(0) \left(1 - q_0 + q_0 e^{\lambda t} \right)^{\frac{1}{\lambda}} e^{(m - \lambda) t}$$

Integrating one more time, we get

$$y_t = y(0) + \dot{y}(0) \int_0^t \left(1 - q_0 + q_0 e^{\lambda s} \right)^{\frac{1}{\lambda}} e^{(m - \lambda) s} ds.$$

Using the definition of v_t and y_t, we have

$$\dot{y}_0 = -v_0 y(0) = my(0)$$

so

$$y_t = y(0)\left(1 + m\int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds\right).$$

Using the definition of v_t we get

$$v_t = -\frac{m\left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)t}}{1 + m\int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds}$$

so

$$\pi_t^u = \frac{\left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}-1} e^{(m-\lambda)t}}{1 + m\int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds} \tag{34}$$

Thus, substituting (34) in the definition for q_t, we get

$$
\begin{aligned}
q_t &= \pi_t^g + q_0 \pi_t^u \\
&= q_0\left(e^{\lambda t} - 1\right)\pi_t^u + q_0 \pi_t^u \\
&= q_0 e^{\lambda t} \pi_t^u \\
&= \frac{q_0\left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}-1} e^{mt}}{1 + m\int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds}
\end{aligned}
$$

Because q_t is monotone, the solution for t_b and t_g is unique.

Finally, we examine $t_g - t_b$, which equals

$$\frac{1}{r+\phi}\log\left(\frac{(r+\phi)V_{t_g}^b - (c + \phi\theta R)}{(r+\phi)L - (c + \phi\theta R)}\right).$$

A necessary condition for the equilibrium to be true is $t_g - t_b > 0$. However, if $m = 0$, this is clearly violated because Assumption ?? guarantees $V_{t_g}^b < L$. If $m \to \infty$,

$$V_{t_g}^b \to \frac{c + \phi R}{r + \phi} - \frac{\phi\left(R - F\right)\left(1 - \theta\right)}{r + \phi}$$

so that it exceeds L. Finally, a quick comparative static analysis shows that $\frac{dV_{t_g}^b}{dm} > 0.$ Therefore, there exists a unique m^* so that such an equilibrium exists if and only if $m > m^*$. $\qquad\square$

A.1.4 Proof of Proposition 4

Proof. Next, we provide an explicit solution when the financial constraint is binding. In equilibrium, the constraint $y_t \leq c$ must bind at time t_b. However, depending on the parameters of the problem, the constraint might be slack at time t_g. In particular, we can show that the constraint on $[t_b, t_g]$ is monotonic, that is, there exists t_c such that $y_t^b = c$ on $[t_b, t_c]$ and $y_t^b < c$ on $(t_c, t_g]$. In the case that $t_c > t_g$, the constraint always binds. As in the case in which we ignore the financial constraint, we have that

$$
\begin{aligned}
V_{t_g}^b &= V_{t_g}^g + \frac{\phi R (\theta - 1) + m\frac{\phi(R-F)(\theta-1)}{r+\phi}}{r + \phi + m} \\
&= \frac{c + \phi R}{r + \phi} + \frac{\phi R (\theta - 1) + m\frac{\phi(R-F)(\theta-1)}{r+\phi}}{r + \phi + m}.
\end{aligned}
$$

Next, we solve for V_t^b when $t \in [t_b, t_g]$, with the boundary condition

$$
V_{t_g}^b (r + \phi) V_t^b = \dot{V}_t^b + c + \phi\theta R
$$

$$
\Longrightarrow
$$

$$
V_t^b = \frac{c + \phi R\theta}{r + \phi} + e^{(r+\phi)(t-t_g)} \left[V_{t_g}^b - \frac{c + \phi R\theta}{r + \phi} \right]
$$

At $t = t_c$, the financial constraint exactly binds so that it must be

$$
L + (1 - \beta) \left(V_t^b - L \right) - B_t^b(r) \equiv \frac{c - rF}{r + \phi + m}, \tag{35}
$$

where $B_t^b(r)$ solve

$$
(r + \phi + m) B_t^b(r) = \dot{B}_t^b(r) + rF + \phi\theta F + m \left[L + (1 - \beta) \left(V_t^b - L \right) \right], \qquad t \in (t_c, t_g).
$$

Let's define $Z_t \equiv L + (1 - \beta) \left(V_t^b - L \right) - B_t^b(r)$. Substituting the ODEs for V_t^b and $B_t^b(r)$, and defining the constant

$$
\Gamma_1 \equiv (r + \phi) \beta L + (1 - \beta) (c + \phi\theta R) - (r + \phi\theta) F,
$$

we get the following ODE for Z_t on (t_c, t_g)

$$
(r + \phi + m) Z_t = \dot{Z}_t + \Gamma_1 \qquad t \in (t_c, t_g). \tag{36}
$$

At time t_g we have

$$B_{t_g}^b(r) = \frac{\phi\theta + r + m}{r + \phi + m}F,$$

which means that

$$Z_{t_g} = L + (1 - \beta)\left(V_{t_g}^b - L\right) - B_{t_g}^b.$$

Solving (36) backward in time and combining with equation (35) we get

$$Z_{t_c} = \frac{\Gamma_1}{r + \phi + m} + e^{(r+\phi+m)(t_c-t_g)}\left[Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}\right] = \frac{c - rF}{r + \phi + m}.$$

From here, we can solve for $t_c - t_g$, which is given by

$$t_c - t_g = \frac{1}{r + \phi + m}\log\left(\frac{c - rF - \Gamma_1}{r + \phi + m}\frac{1}{Z_{t_g} - \frac{\Gamma_1}{r+\phi+m}}\right).$$

Next, letting

$$\Gamma_2 \equiv rF + \phi\theta F + m\frac{c - rF}{r + \phi + m}$$

we can find $B_t^b(r)$ for $t \in (t_b, t_c)$ solving the following ODE

$$(r + \phi)B_t^b = \dot{B}_t^b + \Gamma_2, \qquad t \in (t_b, t_c),$$

with initial condition

$$B_{t_b}^b(r) + \frac{c - rF}{r + \phi + m} = L.$$

The solution to the previous ODE at time t_c is

$$B_{t_c}^b(r) = \frac{\Gamma_2}{r + \phi} + e^{(r+\phi)(t_c-t_b)}\left[L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi}\right]. \tag{37}$$

From equation (35), we also know that at t_c

$$B_{t_c}^b(r) = L + (1 - \beta)\left(V_{t_c}^b - L\right) - \frac{c - rF}{r + \phi + m}, \tag{38}$$

where

$$V_{t_c}^b = \frac{c + \phi R\theta}{r + \phi} + \left[V_{t_g}^b - \frac{c + \phi R\theta}{r + \phi}\right]\left[\frac{c - rF - \Gamma_1}{r + \phi + m}\frac{1}{Z_{t_g} - \frac{\Gamma_1}{r+\phi+m}}\right]^{\frac{r+\phi}{r+\phi+m}}.$$

Combining equations (37) and (38) we get

$$\frac{\Gamma_2}{r+\phi} + e^{(r+\phi)(t_c-t_b)}\left[L - \frac{c-rF}{r+\phi+m} - \frac{\Gamma_2}{r+\phi}\right] = \Gamma_3$$

$$\Longrightarrow$$

$$t_c - t_b = \frac{1}{r+\phi}\log\left(\frac{\Gamma_3 - \frac{\Gamma_2}{r+\phi}}{L - \frac{c-rF}{r+\phi+m} - \frac{\Gamma_2}{r+\phi}}\right),$$

where

$$\Gamma_3 \equiv L + (1-\beta)\left(\frac{c+\phi R\theta}{r+\phi} + \left[V_{t_g}^b - \frac{c+\phi R\theta}{r+\phi}\right]\left[\frac{c-rF-\Gamma_1}{r+\phi+m}\frac{1}{Z_{t_g} - \frac{\Gamma}{r+\phi+m}}\right]^{\frac{r+\phi}{r+\phi+m}} - L\right) - \frac{c-rF}{r+\phi+m}$$

Thus, we get that

$$t_b = t_g + \frac{1}{r+\phi+m}\log\left(\frac{c-rF-\Gamma_1}{r+\phi+m}\frac{1}{Z_{t_g} - \frac{\Gamma_1}{r+\phi+m}}\right) - \frac{1}{r+\phi}\log\left(\frac{\Gamma_3 - \frac{\Gamma_2}{r+\phi}}{L - \frac{c-rF}{r+\phi+m} - \frac{\Gamma_2}{r+\phi}}\right).$$

The previous solution can be simplified significantly when, $y_t = c$ for all $t \in [t_b, t_g]$. This happens if

$$\beta L + (1-\beta)V_{t_g}^b > B_{t_g}^b(r) + \frac{c-rF}{r+\phi+m},$$

which reduces to

$$L - \frac{c+(\phi\theta+m)F}{r+\phi+m} + (1-\beta)\left[\frac{c}{r+\phi} + \frac{\phi}{r+\phi}\left((1-\theta)\frac{mF}{r+\phi+m} + \theta R\right) - L\right] > 0$$

In this case, let $B_{\max}^i(t|t_g)$ be the maximum continuation value that a bank can obtain at time t given that the coupon rate after time t is c and the firm switches to market financing after time t_g.

$$B_{\max}^b(t|t_g) = \frac{(c+\phi\theta)F}{r+\phi}\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}\frac{(c+\phi\theta+m)F}{r+\phi+m}$$

In equilibrium, the thresholds $\{t_b, t_g\}$ must be such $B_{\max}^b(t|t_g) \geq L$ for all $t \in (t_b, t_g)$, and in particular

$$B_{\max}^b(t_b|t_g) = \frac{(c+\phi\theta)F}{r+\phi}\left(1 - e^{-(r+\phi)(t_g-t_b)}\right) + e^{-(r+\phi)(t_g-t_b)}\frac{(c+\phi\theta+m)F}{r+\phi+m} = L$$

which implies (25). □

A.1.5 Proof of Proposition 6

Lemma 6. *For any $t \in (t_b, t_g)$,*

$$V_t^u = q_0 V_t^g + (1 - q_0) V_t^b$$

Proof. On $t_a \in (t_b, t_g)$, the continuation value V_t^i solves

$$(r + \phi) V_t^u = \dot{V}_t^u + c + \phi \left[q_0 + (1 - q_0) \theta \right] R$$
$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R$$
$$(r + \phi) V_t^b = \dot{V}_t^b + c + \phi \theta R,$$

which means that

$$V_t^u = \frac{c + \phi \left[q_0 + (1 - q_0) \theta \right] R}{r + \phi} \left(1 - e^{-(r+\phi)(t_g - t)} \right) + e^{-(r+\phi)(t_g - t)} V_{t_g}^u$$
$$V_t^g = \frac{c + \phi R}{r + \phi} \left(1 - e^{-(r+\phi)(t_g - t)} \right) + e^{-(r+\phi)(t_g - t)} V_{t_g}^g$$
$$V_t^b = \frac{c + \phi \theta R}{r + \phi} \left(1 - e^{-(r+\phi)(t_g - t)} \right) + e^{-(r+\phi)(t_g - t)} V_{t_g}^b.$$

From here we get that

$$q_0 V_t^g + (1 - q_0) V_t^b - V_t^u = e^{-(r+\phi)(t_g - t)} \frac{m}{r + \phi + m} \left(q_0 \bar{V}^g + (1 - q_0) \bar{V}^b - \bar{V}^u \right).$$

We have that

$$\bar{V}^u = \frac{c + \phi \left[\bar{q} + (1 - \bar{q}) \theta \right] F}{\delta + \phi} + \frac{\phi [q_0 + (1 - q_0)\theta] (R - F)}{r + \phi}$$
$$\bar{V}^b = \frac{c + \phi \left[\bar{q} + (1 - \bar{q}) \theta \right] F}{\delta + \phi} + \frac{\phi \theta (R - F)}{r + \phi}$$
$$\bar{V}^g = \frac{c + \phi \left[\bar{q} + (1 - \bar{q}) \theta \right] F}{\delta + \phi} + \frac{\phi (R - F)}{r + \phi}$$

From here we get that $q_0 \bar{V}^g + (1 - q_0) \bar{V}^b - \bar{V}^u = 0$, which means that $q_0 V_t^g + (1 - q_0) V_t^b - V_t^u = 0$. $\square$

Beliefs: For $t \in (t_a, t_b)$ beliefs solve

$$\dot{\pi}_t^u = m\pi_t^u \pi_t^b$$
$$\dot{\pi}_t^g = m\pi_t^g \pi_t^b$$
$$\dot{\pi}_t^b = -m\pi_t^b \left(1 - \pi_t^b\right).$$

In particular,

$$\dot{q}_t = mq_t \pi_t^b,$$

so

$$q_t = q_{t_a} e^{m \int_{t_a}^t \pi_s^b ds}$$

Solving for π_t^b we get

$$\pi_t^b = \frac{\pi_{t_a}^b}{\pi_{t_a}^b + (1 - \pi_{t_a}^b)e^{m(t-t_a)}}.$$

We have that

$$
\begin{aligned}
m \int_{t_a}^t \pi_s^b ds &= \int_{t_a}^t \frac{-\dot{\pi}_s^b}{1 - \pi_s^b} ds \\
&= \log(1 - \pi_s^b)\Big|_{t_a}^t \\
&= \log\left(\frac{1 - \pi_t^b}{1 - \pi_{t_a}^b}\right)
\end{aligned}
$$

so

$$
\begin{aligned}
e^{m \int_{t_a}^t \pi_s^b ds} &= \frac{1 - \pi_t^b}{1 - \pi_{t_a}^b} \\
&= \frac{e^{m(t-t_a)}}{\pi_{t_a}^b + (1 - \pi_{t_a}^b)e^{m(t-t_a)}}
\end{aligned}
$$

so

$$q_t = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-m(t-t_a)}} q_{t_a}.$$

To find $\pi_{t_a}^b$, we use equations (32) and (34) to arrive to

$$\pi_{t_a}^b = 1 - \frac{q_{t_a}}{q_0}\left(q_0 + (1 - q_0)e^{-\lambda t_a}\right).$$

Notice that as $t \to \infty$ we have that

$$\frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-m(t-t_a)}} q_{t_a} \to \frac{q_{t_a}}{1 - \pi_{t_a}^b}$$

This limit is greater than $\bar{q}$ if and only if

$$q_{t_a} \geq (1 - \pi_{t_a}^b)\bar{q} = \frac{q_{t_a}}{q_0}\left(q_0 + (1 - q_0)e^{-\lambda t_a}\right)\bar{q}.$$

Thus, we have that

$$t_a \geq \frac{1}{\lambda}\log\left(\frac{\bar{q}}{1-\bar{q}}\frac{1-q_0}{q_0}\right)$$

Optimality Conditions: First, we show that $t_a < t_b$. Suppose that $t_a \in (t_b, t_g)$, then the HJB equation is given by

$$(r + \phi + m)\, B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta]F + m[L + (1 - \beta)(V_t^u - L)]$$
$$(r + \phi + m)\, B_t^g = \dot{B}_t^g + y_t F + \phi F + m[L + (1 - \beta)(V_t^g - L)]$$
$$(r + \phi + m)\, B_t^b = \dot{B}_t^b + y_t F + \phi\theta F + m[L + (1 - \beta)(V_t^b - L)].$$

From here, we get that

$$(r + \phi + m)\, \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)(q_0 V_t^g + (1 - q_0)V_t^b - V_t^u) = \dot{\Gamma}_t,$$

where the second equation follows from Lemma 6. Hence, $\Gamma_{t_g} = 0$ implies that $\Gamma_t = 0$. From here, we get that if $t_a > t_b$, then $\Gamma_t = 0$ for $t \geq t_a$. Which means that $t_a > t_b$ cannot be an equilibrium.

Given $t_a < t_b$, the threshold t_b is given by the condition $q_{t_b} = \bar{q}$. Using the fact that $\Gamma_{t_b} = 0$, we can solve Γ_t backward in time and solve for t_a such that $\Gamma_{t_a} = \psi/\lambda$. Once we have solved for $\{t_a, t_b, t_g\}$ in this way, the only step left is to verify that Γ_t single-crosses ψ/λ from above at time t_a. Consider the regions $t < t_a$, in this region, we have

$$(r + \phi + m)\, B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta]F - \psi + m[L + (1 - \beta)(V_t^u - L)] + \lambda\Gamma_t$$
$$(r + \phi + m)\, B_t^g = \dot{B}_t^g + y_t F + \phi F + m[L + (1 - \beta)(V_t^g - L)]$$
$$(r + \phi + m)\, B_t^b = \dot{B}_t^b + y_t F + \phi\theta F + mL,$$

so

$$(r + \phi + m + \lambda)\, \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)((1 - q_0)L + q_0 V_t^g - V_t^u) + \psi$$

Letting $H_t \equiv (1 - q_0)L + q_0 V_t^g - V_t^u$, and combining the previous ODE with the ODE for Γ_t

on (t_a, t_b) we get

$$(r + \phi + m + \lambda)\Gamma_t = \dot{\Gamma}_t + m(1 - \beta)H_t + \psi, \ t \in (0, t_a)$$
$$(r + \phi + m)\Gamma_t = \dot{\Gamma}_t + m(1 - \beta)H_t, \ t \in (t_a, t_b)$$

Taking left and right limit at t_a we get that $\dot{\Gamma}_{t_a-} = \dot{\Gamma}_{t_a+}$. Let $\Omega_t \equiv \Gamma_t - \psi/\lambda$, and $\eta \equiv (r + \phi + m)\frac{\psi}{\lambda}$, so

$$(r + \phi + m + \lambda)\Omega_t = \dot{\Omega}_t + m(1 - \beta)H_t - \eta, \ t \in (0, t_a)$$
$$(r + \phi + m)\Omega_t = \dot{\Omega}_t + m(1 - \beta)H_t - \eta, \ t \in (t_a, t_b)$$

Differentiating Ω_t, we get

$$(r + \phi + m + \lambda)\dot{\Omega}_t = \ddot{\Omega}_t + m(1 - \beta)\dot{H}_t, \ t \in (0, t_a)$$
$$(r + \phi + m)\dot{\Omega}_t = \ddot{\Omega}_t + m(1 - \beta)\dot{H}_t, \ t \in (t_a, t_b).$$

Suppose that $\dot{H}_t \leq 0$ on $(0, t_b)$, then $\dot{\Omega}_t = 0 \Rightarrow \ddot{\Omega}_t \geq 0$. Hence, $\dot{\Omega}_t$ single crosses 0 from negative to positive. This implies that if $\dot{H}_t \leq 0$, then Ω_t is quasi-convex on $(0, t_b)$, which means that if $\Omega_{t_a} = 0$ and $\Omega_{t_b} < 0$ (which is necessarily the case as $\Gamma_{t_b} = 0$) it must be the case that $\Omega_t \geq 0$ for $t < t_a$ and $\Omega_t \leq 0$ on (t_a, t_b). It is only left to show that $\dot{H}_t \leq 0$. We have

$$(r + \phi + \lambda)H_t = \dot{H}_t + (r + \phi)L - (1 - q_0)(c + \phi\theta R) + \psi - \lambda(1 - q_0)(V_t^b - L), \ t \in (0, t_a)$$
$$(r + \phi)H_t = \dot{H}_t + (r + \phi)L - (1 - q_0)(c + \phi\theta R), \ t \in (t_a, t_b),$$

where $H_{t_b} = (1 - q_0)V_{t_b}^b + q_0 V_{t_b}^g - V_{t_b}^u = 0$. This means that $\dot{H}_{t_b} < 0$. Differentiating the previous equation we get

$$(r + \phi + \lambda)\dot{H}_t = \ddot{H}_t - \lambda(1 - q_0)\dot{V}_t^b, \ t \in (0, t_a)$$
$$(r + \phi)\dot{H}_t = \ddot{H}_t, \ t \in (t_a, t_b).$$

$\dot{V}_t^b \geq 0$ implies that $\dot{H}_t = 0 \Rightarrow \ddot{H}_t \geq 0$. Hence, $\dot{H}_t$ single crosses 0 from negative to positive, so $\dot{H}_{t_b} < 0 \Rightarrow \dot{H}_t < 0, \forall t \in (0, t_b)$.

Computation Equilibrium Next, derive a system of equations for t_a, t_b, t_g. For $t \in (t_a, t_b)$ we have the following ODE for B_t^i

$$(r + \phi + m) B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta]F + m[L + (1 - \beta)(V_t^u - L)]$$

$$(r + \phi + m) B_t^g = \dot{B}_t^g + y_t F + \phi F + m[L + (1 - \beta)(V_t^g - L)]$$

$$(r + \phi + m) B_t^b = \dot{B}_t^b + y_t F + \phi\theta F + mL.$$

Thus, we get

$$(r + \phi + m)\Gamma_t = \dot{\Gamma}_t + m(1 - \beta)((1 - q_0)L + q_0 V_t^g - V_t^u)$$

which yields

$$\Gamma_t = \int_t^{t^b} e^{-(r+\phi+m)(s-t)} m(1 - \beta)((1 - q_0)L + q_0 V_s^g - V_s^u)ds,$$

where we have used the fact that $\Gamma_{t_b} = 0$. Next, we compute the value of V_t^i.

$$V_t^u = \frac{c + \phi\left[q_0 + (1 - q_0)\,\theta\right]R}{r + \phi}(1 - e^{-(r+\phi)(t_g-t)}) + e^{-(r+\phi)(t_g-t)}V_{t_g}^u$$

$$q_0 V_t^g = \frac{q_0 c + q_0 \phi R}{r + \phi}(1 - e^{-(r+\phi)(t_g-t)}) + e^{-(r+\phi)(t_g-t)}q_0 V_{t_g}^g$$

so

$$(1 - q_0)L + q_0 V_t^g - V_t^u = (1 - q_0)\left[L - \frac{c + \phi\theta R}{r + \phi}(1 - e^{-(r+\phi)(t_g-t)})\right] + e^{-(r+\phi)(t_g-t)}(q_0 V_{t_g}^g - V_{t_g}^u)$$

$$= (1 - q_0)\left[L - \frac{c + \phi\theta R}{r + \phi} + e^{-(r+\phi)(t_g-t)}\left(\frac{c + \phi\theta R}{r + \phi} - V_{t_g}^b\right)\right]$$

Thus, we get

$$\Gamma_t = \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m}\left(L - \frac{c + \phi\theta R}{r + \phi}\right)\left(1 - e^{-(r+\phi+m)(t_g-t)}\right) +$$

$$(1 - \beta)(1 - q_0)\left(\frac{c + \phi\theta R}{r + \phi} - V_{t_g}^b\right)e^{-(r+\phi)(t_g-t)}\left(1 - e^{-m(t_g-t)}\right).$$

Substituting $V_{t_g}^b$ we get the following equation for t_a:

$$\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)\left(1-e^{-(r+\phi+m)(t_g-t_a)}\right)+$$

$$(1-\beta)(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi}-\frac{c+\phi\theta R+m\bar{V}^b}{r+\phi+m}\right)e^{-(r+\phi)(t_g-t_a)}\left(1-e^{-m(t_g-t_a)}\right)=\frac{\psi}{\lambda}. \quad (39)$$

From here, we get that $\{t_a, t_b, t_g\}$ solve

$$\bar{q}=\frac{1}{1-\pi_{t_a}^b+\pi_{t_a}^b e^{-m(t-t_a)}}q_{t_a}$$

$$\frac{\psi}{\lambda}=\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)\left(1-e^{-(r+\phi+m)(t_g-t_a)}\right)$$

$$+(1-\beta)(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi}-\frac{c+\phi\theta R+m\bar{V}^b}{r+\phi+m}\right)e^{-(r+\phi)(t_g-t_a)}\left(1-e^{-m(t_g-t_a)}\right)$$

$$t_g=t_b+\frac{1}{r+\phi}\log\left(\frac{m}{r+\phi+m}\frac{(1-\theta)\phi F}{(r+\phi)L-(c+\phi\theta R)}\right)$$

Finally, we need for conditions for the previous equation to be an equilibrium. Let $\bar{t}_a$ be the threshold the first time $q_t=\bar{q}$ in the benchmark model in which $\psi=0$, which is the same as if $t_a=t_b$. On the other hand, let $\underline{t}_a\equiv\frac{1}{\lambda}\log\left(\frac{\bar{q}}{1-\bar{q}}\frac{1-q_0}{q_0}\right)$ in which case the we have that $\inf\{t>t_a:q_t=\bar{q}\}=\infty$.

We have already shown that if $t_a=\bar{t}_a$, we have $\Gamma_{t_a}=0<\psi/\lambda$. Hence, we only need to show that if $t_a=\underline{t}_a$, then $\Gamma_{t_a}>\psi/\lambda$. In this case $t_g=\infty$, which means that

$$\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)\left(1-e^{-(r+\phi+m)(t_g-t_a)}\right)$$

$$+(1-\beta)(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi}-\frac{c+\phi\theta R+m\bar{V}^b}{r+\phi+m}\right)e^{-(r+\phi)(t_g-t_a)}\left(1-e^{-m(t_g-t_a)}\right)=$$

$$\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)$$

Hence, a solution exists if and only if

$$\frac{\psi}{\lambda}<\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)$$

A.1.6 Rollover gains with instantly-maturing debt

Let us write the full version of Lemma 3, including the rollover gains to other types

Lemma 7. *For the instantly-maturing debt, when $t \in (t_b, t_g)$, the entrepreneur receives rollover gains $G^i dt$ where*

$$G^b = \beta \left[\phi\theta R + c - (r+\phi) L\right] - \phi\theta(R-F) - (c-r)F \tag{40a}$$

$$G^g = \beta \left[rF + \phi F - (r+\phi) L\right] - (1-\beta) \left[\phi (R-F) + (c-r)F\right] \tag{40b}$$

$$G^u = \beta \left[rF + \phi (q_0 + (1-q_0)\theta) F - (r+\phi) L\right] \tag{40c}$$
$$- (1-\beta) \left[\phi (q_0 + (1-q_0)\theta) (R-F) + (c-r)F\right].$$

Proof. Nash Bargaining implies $\beta \left(B_t^b - L\right) = (1-\beta) E_t^b$, which further implies $\beta \dot{B}_t^b = (1-\beta) \dot{E}_t^b$. Multiplying Equation (48) by $(1-\beta)$ (27) by β and take their difference:

$$
\begin{aligned}
-\beta (r+\phi) L &= (1-\beta) \left[\phi\theta (R-F) + (c-r)F + G^b\right] - \beta \left[rF + \phi\theta F - G^b\right] \\
\Rightarrow G^b &= \beta \left[\phi\theta R + c - (r+\phi) L\right] - \phi\theta(R-F) - (c-r)F.
\end{aligned}
$$

Repeating the same calculations for uninformed and the good type we get

$$G^u = \beta \left[rF + \phi (q_0 + (1-q_0)\theta) F - (r+\phi) L\right] - (1-\beta) \left[\phi (q_0 + (1-q_0)\theta) (R-F) + (c-r)F\right]$$

$$G^g = \beta \left[rF + \phi F - (r+\phi) L\right] - (1-\beta) \left[\phi (R-F) + (c-r)F\right]$$

$\square$

A.2 Bank and Entrepreneur Value Function

In this subsection, we supplement the details in subsection 3.1.2. Below, we will describe the value function of the entrepreneur and the bank respectively in three different regions.

In the *Market Financing* region (t_g, ∞), the value of the equity depends on the coupon rate determined at the last rollover date before t_g, which we denote by y.

$$E_t^u = \frac{\phi \left[q_0 + (1-q_0) \theta\right] (R-F)}{r + \phi + \lambda + m} + \frac{\lambda \left[q_0 E_t^g + (1-q_0) E_t^b\right]}{r + \phi + \lambda + m} \tag{41a}$$
$$+ \frac{(c-y) + m \left(\bar{D} - F\right) + m \frac{\phi[q_0 + (1-q_0)\theta](R-F)}{r+\phi}}{r + \phi + \lambda + m}$$

$$E_t^g = \frac{\phi (R-F) + (c-y) + m \left(\bar{D} - F\right) + m \frac{\phi(R-F)}{r+\phi}}{r + \phi + m} \tag{41b}$$

$$E_t^b = \frac{\phi\theta (R-F) + (c-y) + m \left(\bar{D} - F\right) + m \frac{\phi\theta(R-F)}{r+\phi}}{r + \phi + m}, \tag{41c}$$

where $\bar{D} = \bar{D}_{\tau_m}$ in (5) evaluated at $q_{\tau_m} = \bar{q}$. The value function of bank satisfy

$$B_t^u = \frac{rF + \phi\left[q_0 + (1 - q_0)\,\theta\right]F + \lambda\left[q_0 B_t^g + (1 - q_0)\,B_t^b\right] + mF}{r + \phi + \lambda + m} \tag{42a}$$

$$B_t^g = F \tag{42b}$$

$$B_t^b = \frac{rF + \phi\theta F + mF}{r + \phi + m}. \tag{42c}$$

Next, we study the bank's and the entrepreneur's value in the other two regions. In general, these values will depend on the coupon rate that they have agreed on so that we will use $E_t^i(y)$ and $B_t^i(y)$ to denote the values at coupon rate is y. Note this coupon payment will continued to be made until either the project matures or the loan matures, that is, until $\tau = \min\{\tau_m, \tau_\phi\}$. Equivalently, we can write the present value of this coupon payment as $T(y) = \frac{y - rF}{r + m + \phi}$ so that $B_t^i(y) = B_t^i(rF) + T(y)$ and $E_t^i(y) = E_t^i(rF) - T(y)$. For the remainder of this subsection, we will use E_t^i and B_t^i for $E_t^i(rF)$ and $B_t^i(rF)$. For type u and g, when $t \in (0, t_g)$,

$$(r + \phi + m)\,B_t^u = \dot{B}_t^u + rF + \phi\left[q_0 + (1 - q_0)\,\theta\right]F + \lambda\left[q_0 B_t^g + (1 - q_0)\,B_t^b - B_t^u\right] \tag{43a}$$

$$+ m\left[L + (1 - \beta)\,(V_t^u - L)\right] \tag{43b}$$

$$(r + \phi + m)\,E_t^u = \dot{E}_t^u + (c - rF) + \phi\left[q_0 + (1 - q_0)\,\theta\right](R - F) + m\beta\,(V_t^u - L) \tag{43c}$$

$$+ \lambda\left[q_0 E_t^g + (1 - q_0)\,E_t^b - E_t^u\right]$$

$$(r + \phi + m)\,B_t^g = \dot{B}_t^g + rF + \phi F + m\left[L + (1 - \beta)\,(V_t^g - L)\right] \tag{43d}$$

$$(r + \phi + m)\,E_t^g = \dot{E}_t^g + (c - rF) + \phi\,(R - F) + m\beta\,(V_t^g - L). \tag{43e}$$

In contrast, the value functions for a bad-type entrepreneur differ across the two regions.

$$(r + \phi + m)\,E_t^b = \dot{E}_t^b + (c - rF) + \phi\theta\,(R - F) \quad \forall t \in (0, t_b) \tag{44a}$$

$$(r + \phi + m)\,B_t^b = \dot{B}_t^b + rF + \phi\theta F + mL \tag{44b}$$

$$(r + \phi + m)\,E_t^b = \dot{E}_t^b + (c - rF) + \phi\theta\,(R - F) + m\beta\,(V_t^b - L) \quad \forall t \in (t_b, t_g) \tag{44c}$$

$$(r + \phi + m)\,B_t^b = \dot{B}_t^b + rF + \phi\theta F + m\left[L + (1 - \beta)\,(V_t^b - L)\right]. \tag{44d}$$

Intuitively, in the *efficient liquidation* region, a bad project gets liquidated when the loan matures, whereas in the *zombie lending* region, the same loan will get rolled over.

Finally, given that we have shown the value function E_t^b can be non-monotonic in $[t_b, t_g)$, Lemma 8 proves that in the region of (t_b, t_g), $\dot{E}_t^b$ will change sign at most once. Therefore,

the value of E_t^b is either monotonically increasing, or first increases and then decreases.

Lemma 8. $\ddot{E}_t^b > 0$ *for* $t \in (t_b, t_g)$.

Proof. Take derivative to both sides of equation (44c), we can get

$$\ddot{E}_t^b = (r + \phi + m)\,\dot{E}_t^b - m\beta\dot{V}_t^b.$$

This implies any local extrema of E_t^b (which satisfies $\dot{E}_t^b = 0$) is a local maximum. if $\dot{V}_t^b > 0$. Therefore, if $\dot{V}_t^b > 0$ for any $t \in (t_b, t_g)$, E_t^b cannot change sign more than once over $t \in (t_b, t_g)$. To show this, let us take derivative to both sides of equation (18c) in region $t \in [t_b, t_g)$

$$\ddot{V}_t^b = (r + \phi)\,\dot{V}_t^b.$$

At $t = t_b$, $\dot{V}_{t_b}^b = (r + \phi)\,L - c - \phi\theta R > 0$ following Assumption 1.Therefore, since sign $\left(\dot{V}_t^b\right) =$ sign $\left(\ddot{V}_t^b\right)$ for any $t \in (t_b, t_g)$, that implies $\dot{V}_t^b > 0$ in this region as well.

□

A.3 Instantly-Maturing Debt

When debt is rolled over in a continuous basis, at any time $t < t_b$ and as long as the project has not been liquidated, the market knows that the bank has not received bad news. In this case, the solution for q_t simplifies significantly, and the solution in (16) converges to

$$q_t = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t}}.$$

Thus, we can solve for the threshold t_b

$$t_b = \frac{1}{\lambda}\left[\log\left(\frac{\bar{q}}{1 - \bar{q}}\right) - \log\left(\frac{q_0}{1 - q_0}\right)\right] \tag{45a}$$

$$t_g = t_b + \frac{1}{r + \phi}\log\left(\frac{\phi(1 - \theta)F}{(r + \phi)L - rF(1 + \gamma) - \phi\theta R}\right). \tag{45b}$$

We can also solve for the firm value in closed form. For $t \leq t_b$, the HJB equation reduces to

$$(r + \phi + \lambda)\,V_t^u = \dot{V}_t^u + c + \phi\left[q_0 + (1 - q_0)\,\theta\right]R + \lambda\left[q_0 V_t^g + (1 - q_0)\,L\right] \tag{46a}$$

$$(r + \phi)\,V_t^g = \dot{V}_t^g + c + \phi R \tag{46b}$$

$$V_t^b = L. \tag{46c}$$

For $t \in (t_b, t_g)$, the equations are unchanged, except that the informed-bad type's becomes

$$(r + \phi) V_t^b = \dot{V}_t^b + c + \phi \theta R. \tag{47}$$

With instant maturing debt, the entrepreneur refinances with market-based lenders right upon time t reaches t_g, in which case the valuation equations are identical to (4). This equations can be solved in closed form. For $t < t_b$, the value of the good and uninformed firm are

$$V_t^u = \bar{V}^u e^{-(r+\phi+\lambda)(t_g-t)} + q_0 \bar{V}^g \left(1 - e^{-\lambda(t_g-t)} \right) e^{-(r+\phi)(t_g-t)}$$

$$+ \left[\frac{\lambda}{(r+\phi)(r+\phi+\lambda)} + \frac{e^{-(r+\phi+\lambda)(t_g-t)}}{r+\phi+\lambda} - \frac{e^{-(r+\phi)(t_g-t)}}{r+\phi} \right] q_0 \left(c + \phi R \right)$$

$$+ \frac{1 - e^{-(r+\phi+\lambda)(t_g-t)}}{r+\phi+\lambda} \left(c + \phi \left[q_0 + (1-q_0) \theta \right] R + \lambda(1-q_0)L \right)$$

$$V_t^g = \frac{1 - e^{-(r+\phi)(t_g-t)}}{r+\phi} (c + \phi R) + e^{-(r+\phi)(t_g-t)} \bar{V}^g$$

Next, we supplement the HJBs of bank when $t \in (t_b, t_g)$.

$$(r + \phi) B_t^b = \dot{B}_t^b + rF + \phi \theta F - G_t^b. \tag{48}$$

A.4 Tax Shields